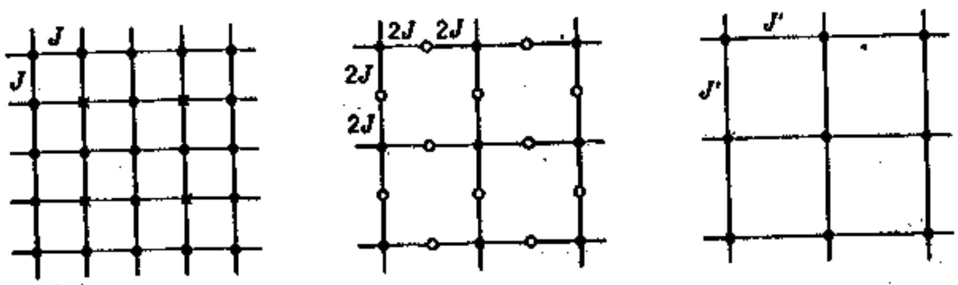
Migdal-Kadanoff approximation. Consider a two-dimensional Ising model on a square lattice without external field:

$$E(s) = -J \sum_{\langle i,j \rangle} s_i s_j$$

The Migdal-Kadanoff approximation consists of two steps, illustrated in the sketches. First, the sites marked by an $\times$ in sketch (a) are removed. To compensate for this, the remaining bonds are doubled in strength, so that the Hamiltonian becomes

$$E(s) = -2J \sum_{\langle i,j \rangle}' s_i s_j$$

where the prime indicates that this Hamiltonian is defined on the lattice shown in sketch



(a) (b) (c)

(a) Make a renormalization-group transformation by summing over the spins on the sites indicated by open circles in sketch (b), to end up with the lattice of sketch (c). Show that one obtains a nearest-neighbor Ising model on the new lattice, with Hamiltonian

$$E'(s) = -J' \sum_{\langle i,j \rangle} s_i s_j$$

such that the partition differs from the original one by only a constant factor. Put $kT = 1$. Defining $x = 2J$, $x' = 2J'$, derive the recursion relation

$$x' = \tfrac{1}{2}(x^2 + x^{-2})$$

(b) Recall that in the one-dimensional Ising model the first points are trivial ones at $x^* = 0$ and $x^* = 1$. Show that in addition to these, there is a new fixed point. Expanding x' in its neighborhood, show

$$x' - x^* = \left(x^* - \frac{1}{x^{*3}}\right)(x - x^*)$$

Show that the new fixed point is unstable.

(c) Calculate the exponent ν in terms of x^*.