1 Homework 2, due February 19, 2020

The Hamiltonian of this problem is the nearest neighbor Ising model

$$H = -\frac{\kappa}{2}\sum_{\langle ij\rangle} S_i S_j - H\sum_i S_i, \tag{1}$$

where the sums run over all N spins.
a) Explain that you cannot just use the Hubbard-Statonovich transformation as you did last week, and argue that you need a redifinition of the zero of energy.
b) Show that

$$\int_{-\infty}^{\infty}\prod_i \frac{dx_i}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i A_{ij} x_j + x_i B_i} = \det{}^{-1/2} e^{\frac{1}{2}B_i A_{ij}^{-1} B_j} \tag{2}$$

with A a real symmetric positive definite matrix and B a vector.
c) Use this identity to show that

$$Z = \int_{-\infty}^{\infty}\prod_k d\chi_k d^{-\beta S} \tag{3}$$

with

$$S = \frac{1}{2}(\chi_k - H_k)J_{kl}^{-1})(\chi_l - H_l) - \frac{1}{\beta}\sum_k \log(2\cosh\beta\chi_k). \tag{4}$$

d) Calculate the integral by a saddle point approximation with solution $\bar{\chi}_k$ and show that the magnetization at site k is given by

$$m_k = \tanh\beta\chi_k. \tag{5}$$

e) Show that the mean field value of S is given by

$$\bar{S} = \frac{1}{2}J - i m_i m_j - \frac{1}{\beta}\sum_i \log(1/\sqrt{1-m_i^2}). \tag{6}$$

f) Calculate the mean field approximation to the Gibbs free energy from the Legendre transform

$$\Gamma(m_i) = \bar{S} + \sum_i H_i(m_i) m_i, \tag{7}$$

and show that the equation of state is given by

$$H_i = \frac{\partial\Gamma}{\partial m_i}. \tag{8}$$