Linear differential equations

Homogeneous differential equation

$$L y = 0 \qquad L y = P_0(x) y^{(n)} + \cdots + P_n(x)\, y$$

n'th order diff. eq.

Inhomogeneous differential eqs

n'th order diff. eq. is equivalent to a set of n first order equation

$$y_1 = y \qquad \text{then} \qquad y_2 = y_1'$$
$$y_2 = y' \qquad\qquad y_3 = y_2'$$
$$y_3 = y'' \qquad\qquad \vdots$$
$$\vdots$$
$$y_n = y^{(n-1)} \qquad \frac{d}{dx} y_n = \frac{-1}{P_0}\left(P_1 y_{n-1} + \cdots P_n y\right)$$

Flow equations $\quad X^i(x^1, x^2, \ldots x^n, t)$
vector field.

$$\frac{d x^i}{d t} = X^i(x_1, \ldots x^n, t)$$

initial condition $X^i(t=0) = X^i_0$

It is clear that there is a unique solution

The first order flow equations can also be combined in a single n'th order equation in the t derivatives

The solutions of the n'th order equation are a vector space

y_1, y_2 solutions $\Rightarrow a y_1 + b y_2$ is a solution

There are n independent solutions

consider solutions with the initial conditions

1) $y_1(0) = 1 \qquad y_1^{(k)}(0) = 0 \quad k > 1$

2) $y_2^{(2)}(0) = 1 \qquad y_2^{(k \neq 2)}(0) = 0$

$\vdots$

n) $y_n^{(n)}(0) = 1 \qquad y_n^{(k \neq n)}(0) = 0$

now if they are not linear independent

then $\exists \quad \lambda_k : \lambda_1 y_1 + \cdots \lambda_n y_n = 0$

$\Rightarrow \lambda_1 = 0$

also any derivative $\lambda_1 y_1^{(p)} + \cdots + \lambda_n y_n^{(p)} = 0$

$\Rightarrow \lambda_p = 0$

$\Rightarrow$ all $\lambda_k = 0 \Rightarrow y_1, \ldots, y_n$ are linearly independent

Wronskian

To find out if solutions are linearly independent, we calculate the Wronskian

$$W(y_1, \ldots y_n; x) = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ y_1^{(2)} & \cdots & y_n^{(2)} \\ \vdots & & \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Differenting a determinant is differentiating each row or column.

$\Rightarrow$

$$\frac{dW}{dx} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

all other terms vanish.

and $y_k^{(n)} = -\frac{P_1}{P_0} y_k^{(n-1)} - \cdots - \frac{P_n}{n} y_k$

only the term $-\frac{P_1}{P_0} y_k^{(n-1)}$ gives a nonzero contribution

$$\Rightarrow \frac{dW}{dx} = -\frac{P_1}{P_0} W$$

$$\Rightarrow W(x) = W(x_0)\, e^{-\int_{x_0}^{x} \frac{P_1(x')}{P_0(x')} dx'}$$

Liouville's theorem

W vanishes at all x or never because an exponential function is always nonzero.

Theorem: The Wronskian vanishes for linearly dependent function.

$$\lambda_1 y_1(x) + \cdots + \lambda_n y_n(x) = 0$$

$$\Rightarrow \lambda_1 y_1^{(n)}(x) + \cdots + \lambda_n y_n^{(n)}(x) = 0$$

This has a nontrivial solution

$$\Rightarrow W = 0$$

Converse $y_1 \dots y_n$ solutions of an n'th order ODE with $W(y_k, x_0) = 0$

Then $\exists \lambda_k$ not all zero such that $\lambda_1 y_1(x) + \cdots + \lambda_n y_n(x) \equiv Y(x)$

and $Y(x_0) = Y'(x_0) = \cdots = Y^{(n-1)}(x_0) = 0$

These are linear eqs. for the λ_k and have a nontrivial solution if $W(y_k, x_0) = 0$

So Y(x) is a solution of an ODE with vanishing initial condition —> Y(x) =0 and y_k are linearly dependent
If we have n solutions of ODE with vanishing Wronskian, then these solutions are linearly dependent.

Normal Form

For polynomials $a_0 x^n + a_1 x^{n-1} + \cdots + a_n$

it is in normal form if $a_1 = 0$

This is achieved by $x \to x - \left(\frac{a_1}{a_0}\right)\frac{1}{n}$

For differential equations

$$P_0 y^{(n)} + P_1 y^{(n-1)} + \cdots P_n y = 0$$

the normal form is when $P_1 = 0$

To find it we substitute $y = w\hat{y}$

then $y^{(n)} = w\hat{y}^{(n)} + n w' \hat{y}^{(n-1)} + \binom{n}{2} w'' \hat{y}^{(n-2)}$

So we require $P_0 n w' + P_1 = 0$

$$\Rightarrow w' = -\frac{P_1}{n P_0}$$

$$\Rightarrow w(x) = w(0)\, e^{-\frac{1}{n}\int_0^x \frac{P_1(x')}{P_1(x')}dx'}$$

normal form of second order equation

$$P_0 y'' + P_2 = 0 \Rightarrow y'' + \frac{P_2}{P_0} y = 0$$

This is like a Schrödinger equation

Inhomogeneous equations

$$P_0(x)\, y^{(n)} + P_1(x)\, y^{(n-1)} + \dots + P_n(x) = f(x)$$

Guess a solution y_G

Then general solution is $y_G + y_H$

with y_H is solution of the Homogeneous equation (also called complementary function)

$$P_0(x)\, y_H^{(n)} + \dots + P_n(x) = 0 .$$

Example

$$R \frac{dQ}{dt} + \frac{Q}{C} = V$$

Homogeneous equation $\quad R \frac{dQ}{dt} + \frac{Q}{C} = 0$

$$\Rightarrow \quad Q = c_1 e^{-\frac{t}{RC}}$$

$Q = CV$ is a solution of the inhomogeneous equation

$\Rightarrow$ general solution is $Q(t) = CV + c_1 e^{-\frac{t}{RC}}$

initial conditio $Q(t_0) = 0 \quad \Rightarrow c_1 = -CV$

$$\Rightarrow \quad Q(t) = CV\left(1 - e^{-\frac{t}{RC}}\right)$$

Variation of parameters

This is a trick to solve the inhomogeneous equation invented by Lagrange

We start from a solution of the homogeneous equation

$$y = \upsilon_1 y_1 + \cdots + \upsilon_n y_n$$

the υ_k are functions of x and the y_k are independent solutions.

$$y' = \upsilon_1 y_1' + \cdots + \upsilon_n y_n' + (\underbrace{\upsilon_1' y_1 + \upsilon_2' y_2 + \cdots \upsilon_n' y_n}_{\text{choose } \upsilon_k' \text{ such that this vanish}})$$

then $y'' = \upsilon_1 y_1'' + \cdots + \upsilon_n y_n'' + (\underbrace{\upsilon_1' y_1' + \cdots + \upsilon_n' y_n'}_{\text{choose } \upsilon_k' \text{ such that this vanishes.}})$

continues $y^{(n)} = \upsilon_1 y_1^{(n)} + \cdots + \upsilon_n y_n^{(n)} +$

$$+\left(\upsilon_1' y_1^{(n-1)} + \upsilon_2' y_2^{(n-1)} + \cdots + \upsilon_n' y_n^{(n-1)} = \frac{F(x)}{P_0(x)}\right)$$

$\Rightarrow$ $P_0 y^{(n)} + P_1 y^{(n-1)} + \cdots + y =$

$P_0 (\upsilon_1 y_1^{(n)} + \upsilon_2 y_2^{(n)} + \cdots) \quad + F(x)$

$P_1 (\upsilon_1 y_1^{(n-1)} + \upsilon_2 y_2^{n-1} + \cdots)$

$\underbrace{+ \; P_n (\upsilon_1 y_1 + \upsilon_2 y_2}_{0}$ $\Rightarrow$ this is a solution of the inhomogeneous diff. eq we still have to solve for the v_k

$$v_1' y_1 + \dots v_n' y_n = 0$$

$$\vdots$$

$$v_1' y_1^{(n-1)} \dots + v_n' y^{(n-1)} = \frac{f(x)}{p_0}$$

linear equations
they have a solution if $\begin{vmatrix} y_1 & \dots & y_n \\ \vdots & & \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0$

This is the case because we assumed that the solutions are independent.

<u>Example</u> $\quad \frac{dy}{dx} + P(x)\, y = f(x)$

homogeneous equation $\quad \frac{dy}{dx} + P(x)\, y = 0$

solution $\quad y_1 = e^{-\int_a^x P(s)\, ds}$

substitute $y_1 v_1$

$$y_1' v_1 + P(x)\, y_1 + y_1 v' = f(x)$$

require that $\quad y_1 v' = f(x)$

$$\Rightarrow v' = \frac{f(x)}{y_1} \Rightarrow v = e^{-\int_b^x \frac{f(x')}{y_1} dx'}$$

then $y_1 v_1$ is solution of inhomogeneous equation.

Singular point:

$$P_0(x)\, y^{(n)} + P_1(x)\, y^{(n-1)} + \cdots$$

singular point if $P_0(a) = 0$

This completely change the solutions
It often happens that P_0 vanishes at the end of an interval.
Generally this is a complicated subject and we will only look at a seconde order example:

$$P_0 y'' + P_1 y' + P_2 = 0$$

$$P_0(x) = (x-a)^2 P(x)$$
$$P_1(x) = (x-a)\, Q(x)$$
$$P_2(x) = R(x)$$

P, Q R smooth and nonzero P, Q

$x = a$ is a regular singular point

close to $x = a$ the equation can be approximated by

$$P(a)(x-a)^2 y'' + Q(a)(x-a) y' + R(a) y$$

solution $y = (x-a)^\lambda$

$$P(a)\, \lambda(\lambda - 1) + Q(a)\, \lambda + R(a) = 0$$

Two solutions $y_1 = (x-a)^{\lambda_1} f_1(x)$

$y_2 = (x-a)^{\lambda_2} f_2(x)$

By substituting this back into the differential equation we get equations for f_1 and f_2