

Function spaces

$C^n [a, b]$ n fold differentiable functions on $[a, b]$

$C^\infty [a, b]$ Analytic function on $[a, b]$

A function space is a linear vector space

$\lambda f(x) + \mu g(x)$ is also a function

To find the distance between two functions we need a norm. There

are many different norms, eg

$$\int_a^b |f(x)| dx$$

A norm should satisfy:

i) positivity $\|f\| \geq 0$, $\|f\| = 0$ implies $f = 0$

ii) triangle inequality

$$\|f + g\| \leq \|f\| + \|g\|$$

iii) linear homogeneity

$$\|\lambda f\| = |\lambda| \|f\|$$

i) $\int_a^b |f(x)| dx$ satisfies 1)

$$\text{ii) } \int_a^b |f(x) + g(x)| dx \leq \int_a^b (|f(x)| + |g(x)|) dx = \|f\| + \|g\|$$

iii) is obvious

Convergence

We often have to approximate a function $f(x)$ by a sequence $f_n(x)$ on D

There are different ways of converging.

i) Pointwise convergence: $\forall x \in D$, the sequence $f_n(x)$ converges to $f(x)$

ii) Uniform convergence

$$\sup_{x \in D} |f_n(x) - f(x)| \rightarrow 0 \text{ for } n \rightarrow \infty$$

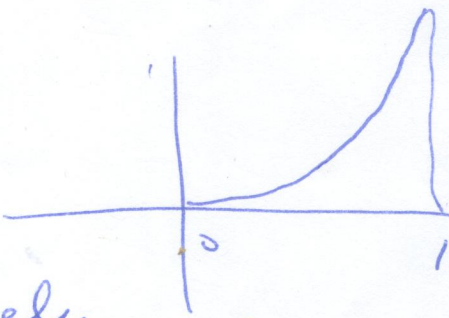
iii) Convergence in the mean

$$\int_D |f_n(x) - f(x)| dx \rightarrow 0 \text{ for } n \rightarrow \infty$$

Uniform convergence is important because it is a condition for interchanging integrals and limits.

Example 1) $f_n = x^n$

- $f_n \rightarrow 0$ pointwise on $[0, 1)$
- does not converge uniformly

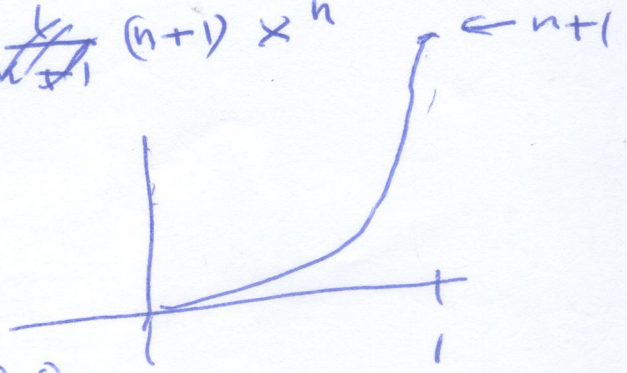


one $[0, 1]$ is does also not converge pointwise
but converges in the mean

Example 2

$$f_n(x) = \frac{1}{n+1} (n+1) x^n \leftarrow n+1$$

$$\int_0^1 f_n(x) dx = 1$$



- converges pointwise to 0
on $[0, 1)$

- does not converge uniformly on $[0, 1)$

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = 1$$

$$\int_0^1 \lim_{n \rightarrow \infty} f_n(x) dx = 0$$

Almost all notion

With convergence in the Lebesgue sense
a function converges to a function
in almost all points, i.e. $\forall x \in D$
with the exception of a set of zero
measure.

norm for Lebesgue space

L^p norm

$$\|f\|_p = \left(\int |f(x)|^p dx \right)^{1/p}$$

Most common norm $p=2$

Cauchy sequence

$\forall \epsilon > 0 \quad \exists N$ such that $\forall n, m > N \quad \|f_n - f_m\| < \epsilon$

a normed vector space is complete if each Cauchy sequence converges to some element of the space.

Banach space a normed complete vector space

$L^p[a, b]$ space are complete when the norm is interpreted as Lebesgue integral

Hilbert space Banach space $L^2[a, b]$

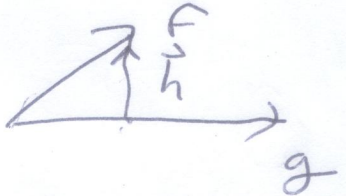
in general, a Hilbert space is a Banach space where the norm is derived from the inner product.

For $L^2[a, b]$

$$\begin{aligned} \|f\| &= \left(\int_a^b f(x) dx \right)^{1/2} \\ &= (f, f)^{1/2} \end{aligned}$$

proof that $\|f\|_2$ is a norm

Cauchy - Schwartz inequality

$$|(f, g)| \leq \|f\| \|g\|$$


define $h = f - \frac{(f, g)}{(g, g)} g$

then $(h, g) = 0 \Rightarrow h \perp g$

apply Pythagoras to h and

$$\|f\|^2 = \left\| \frac{(f, g)}{(g, g)} g \right\|^2 + \frac{|(f, g)|^2 \|g\|^2}{\|g\|^4}$$

$$\Rightarrow |(f, g)|^2 \leq \|f\|^2 \|g\|^2$$

Triangle inequality

$$\|f+g\|^2 \leq \|f\|^2 + \|g\|^2 + 2(f, g)$$

$$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\|$$

$$= (\|f\| + \|g\|)^2$$

of course $\|f\| \geq 0$

$$\text{and } \|\lambda f\| = |\lambda| \|f\|$$

and if $\|f\| = 0$ then $f = 0$

Consequence of Cauchy-Schwarz inequality

$$\|f_n - f\| \rightarrow 0$$

$$\text{then } |(f_n, g) - (f, g)| \\ = |(f_n - f, g)| \leq \|f_n - f\| \|g\|$$

$$\Rightarrow (f_n, g) \rightarrow (f, g)$$

Orthonormal function sets

$$(u_n, u_m) = \delta_{nm}$$

Example a) $u_n = e^{2\pi i n x}$

$$\text{then } \int_a^b dx (u_n, u_m) = \delta_{nm}$$

↑ complex inner product

b) $u_m = \sqrt{2} \sin m \pi x$

if u_n is a complete set on (a, b) then any function f in the Hilbert space can be expanded as

$$f = \sum_{n=0}^{\infty} a_n u_n(x)$$

$$(u_n, f) = \sum_{m=0}^{\infty} a_m (u_n, u_m) = a_n$$

Orthogonal polynomials

inner product $(u, v) = \int_a^b w(x) u(x) v(x) dx$

polynomials depend on w .

$$P_0(x) = \frac{1}{\|1\|_w}$$

$$P_n(x) = \sum_{k=0}^n a_k x^k$$

orthonormal polynomials $(P_n, P_m) = \delta_{nm}$

Construction

$$P_{n+1} = \alpha x P_n + \sum_{k=0}^n a_k P_k$$

$$(P_{n+1}, P_k) = 0 \quad \forall k \leq n$$

$$k \leq n \quad \alpha (P_k, x P_n) + a_k = 0$$

$$\Rightarrow P_{n+1} = \alpha x P_n + \sum_{k=0}^n (-\alpha) (P_k, x P_n) P_k$$

α follows from normalization of

P_{n+1}

$$(P_{n+1}, P_{n+1}) = 1 = \alpha^2 (x P_n, x P_n) - 2\alpha^2 \sum_{k=0}^n (P_k, x P_n) + \sum_{k=0}^n \alpha^2 (P_k, x P_n)^2$$

Best Approximation

$$\Delta = \left\| f - \sum_{n=1}^N a_n u_n \right\|^2$$

← orthonormal basis

we minimize Δ

$$\Delta = \|f\|^2 - \sum_n a_n (f, u_n) - \sum a_n^* (u_n, f) + \sum a_m^* a_n \underbrace{(u_m, u_n)}_{\delta_{mn}}$$

\uparrow
 $\sum a_n^2$

$$\Rightarrow \Delta = \|f\|^2 + \sum_{n=1}^N |a_n - (u_n, f)|^2 - \sum_{n=1}^N |(f, u_n)|^2$$

this is minimized if $a_n = (u_n, f)$

Parseval's theorem

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

$$\|f\|^2 = \int f^*(x) f(x) dx$$

$$= \sum_{nm} \int a_n^* a_m u_n^* u_m dx$$

$$= \sum |a_n|^2$$

Example $u_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ on $[-\pi, \pi]$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{iyx}$$

$$\|f\| = \int_{-\pi}^{\pi} \frac{1}{2\pi} dx = 1$$

$$\frac{1}{\sqrt{2\pi}} e^{iyx} = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iyx - inx} dx = \frac{\sin \pi(y-n)}{\pi(y-n)}$$

Parseval : $\sum_n \frac{\sin^2 \pi(y-n)}{(\pi(y-n))^2} = 1$

$$\Rightarrow \sum_n \frac{\sin^2 \pi y}{(\pi(y-n))^2} = 1$$

$$\Rightarrow \sum_n \frac{1}{\pi^2 (y-n)^2} = \frac{1}{\sin^2 \pi y}$$

Three step recursion relation

Orthogonal polynomials satisfy a 3 step recursion relation

$$x P_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x)$$

Proof $(x P_n, P_k) = (P_n, x P_k) = 0$
 for $k \leq n-2$
 \Rightarrow qed

Weierstrass approximation theorem

$f(x)$ on $[a, b]$ continuous

then \exists polynomial $P(x)$ such

that $|f(x) - P(x)| < \epsilon \quad \forall x \in [a, b]$

\Rightarrow Orthogonal polynomials are complete

So any function can be expressed in terms of an (infinite) sum of orthogonal polynomials.

Example of orthogonal polynomials

Legendre polynomials

$$[-1, 1] \quad w(x) = 1$$

The Legendre polynomial satisfies the Rodriguez formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Proof i) P_n is a polynomial of order n

(ii) orthogonality $k < n$

$$\int_{-1}^1 \frac{1}{2^n n!} \frac{1}{2^k k!} \frac{d^n}{dx^n} (x^2 - 1)^n \frac{d^k}{dx^k} (x^2 - 1)^k$$

- partial integrate $k+1$
- boundary terms vanish because of $(x^2 - 1)^k$

$$= 0 \text{ for } n > k$$

for $n = k$ we get

$$\frac{(2n)!}{2^{2n} n! n!} \int_{-1}^1 (x^2 - 1)^n (-1)^n dx = \frac{2}{2n+1}$$

Recursion

$$(2n+1) x P_n = (n+1) P_{n+1} + n P_{n-1}$$

example $P_0 = 1$

$$P_1 = ax + b$$

$$\int_{-1}^1 (ax+b) P_0 = 0 \quad b + a \frac{1}{2} \cdot 0 = 0$$
$$\Rightarrow b = 0$$

$$\int_{-1}^1 (ax)^2 dx = \frac{2}{3} = a^2 \frac{1}{3} \cdot 2 \Rightarrow a = \sqrt{\frac{3}{2}}$$
$$a = 1$$

$$P_2 = ax^2 + bx + c$$

can do the same but simpler to use Rodriguez formula

$$P_2(x) = \frac{1}{4 \cdot 2} \frac{d^2}{dx^2} (x^2 - 1)^2$$
$$= \frac{1}{8} (12x^2 - 4) = \frac{3}{2} x^2 - \frac{1}{2}$$

Let us see if recursion work

$n=1$

$$3 \times P_1 = 2 P_2 + P_0$$

$$3 \times x = (6x^2 - 1) + 1$$

Hermite polynomials

(54)

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx \sim \delta_{nm}$$

We will see that this follows from:

Generating Function $\sum_{n=0}^{\infty} H_n \frac{t^n}{n!} = e^{2tx - t^2}$

From generating function

$$H_n(x) = \left. \frac{d^n}{dt^n} e^{2tx - t^2} \right|_{t=0}$$

$$= \left. \frac{d^n}{dt^n} e^{-(t-x)^2 + x^2} \right|_{t=0}$$

$$= (-1)^n \left[\left. \frac{d^n}{dx^n} e^{-(t-x)^2} \right]_{x^2} \right|_{t=0}$$

$$= (-1)^n e^{x^2} \left. \frac{d^n}{dx^n} e^{-x^2} \right|_{t=0}$$

Orthogonality relation

$$\int_{-\infty}^{\infty} H_n H_m e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

proof choose $n \neq m$

$$\int_{-\infty}^{\infty} H_{n+m} e^{-x^2} x^m \frac{d^n}{dx^n} e^{-x^2} e^{x^2} \frac{d^m}{dx^m} e^{-x^2}$$

mth order polynomial

so the integral vanishes by partial integration.

for $n = m$: $\int dx \left(\frac{d^n}{dx^n} e^{-x^2} \right) e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$
nth order polynomial

highest power is obtained by differentiating the exponent n times

$$= \int dx \left(\frac{d^n}{dx^n} e^{-x^2} \right) \left((2x)^n + \text{lower order} \right)$$

partially integrate n times

$$= \int dx e^{-x^2} 2^n n!$$

$$= 2^n n! \sqrt{\pi}$$

Chebyshev polynomials

(56)

First kind $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} U_n(x) U_m(x) \propto \delta_{nm}$

Second kind $\int_{-1}^1 dx \sqrt{1-x^2} T_n(x) T_m(x) \propto \delta_{nm}$

They follow from the orthogonality of $\cos m\theta$ and $\sin n\theta$

$$\int_0^\pi \cos n\theta \cos m\theta d\theta \propto \delta_{nm}$$

$$x = \cos \theta \quad dx = -\sin \theta d\theta \\ = -\sqrt{1-x^2} d\theta$$

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \underbrace{\cos^n(\cos^{-1}x)}_{T_n(x)} \cos m(\cos^{-1}x) \propto \delta_{nm}$$

$T_n(x)$ is polynomial because $\cos^n x = \sum_{k=0}^n a_k \cos^k x$

$$\int_0^\pi d\theta \sin n\theta \sin m\theta \propto \delta_{nm}$$

$$= \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} \underbrace{\sin(n \cos^{-1}x)}_{\text{not a polynomial}} \sin(m \cos^{-1}x)$$

$$\sin \cos^{-1}x = \sqrt{1 - \cos^2 \cos^{-1}x} = \sqrt{1-x^2}$$

$$= \int_{-1}^1 dx \sqrt{1-x^2} \frac{\sin(n \cos^{-1}x)}{\sqrt{1-x^2}} \frac{\sin(m \cos^{-1}x)}{\sqrt{1-x^2}}$$

$\sin nx = \sum_{k=0}^n b_k \sin^k x \Rightarrow \underbrace{\frac{\sin(n \cos^{-1}x)}{\sqrt{1-x^2}}}_{U_n(x)} \text{ is polynomial}$

Distribution and test functions

Linear operator in function space

$$f(x) = \int_a^b A(x, y) g(y) dy$$

δ - function

$$f(x) = \int \delta(x-y) f(y) dy$$

↑ like identity operator

$\delta(x-y)$ is called a distribution and f is a test function which is a collection of smooth but otherwise undetermined functions

Derivative of δ function, $\delta'(x-y)$ which is also a linear operator on function space

$$\int \delta'(x-y) f(y) dy$$

$$= \int (-1) \delta(x-y) f'(y) dy = f'(x)$$

↑ part-int.

$$\Rightarrow \delta'(x-y) = \delta(x-y) \frac{d}{dy}$$

$$(n) = d^n/dx^n$$

(50)

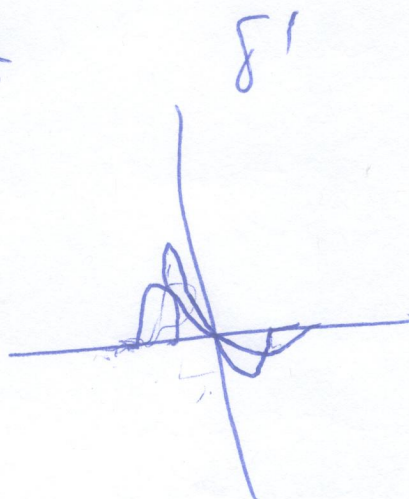
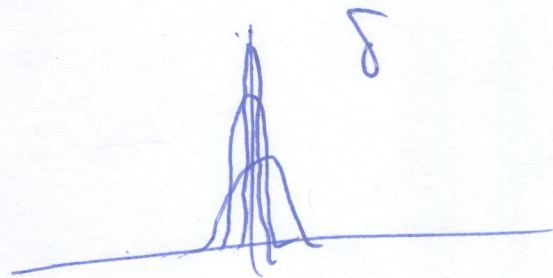
in general $\delta^{(n)}(x-y) = (-1)^n \delta(x-y) \frac{d^n}{dy^n}$

$$\delta'(-y) = -\delta'(y)$$

$$\delta^{(n)}(-y) = (-1)^n \delta^{(n)}(y)$$

$\delta(y)$ is even $\Rightarrow \delta'(y)$ is odd.

δ function as limit



Distributions are functions with discontinuities and derivatives thereof. They are defined by acting with them on a set of nice functions, called test functions e.g. infinitely differentiable functions that vanish sufficiently fast at ∞ .

Distributions are in the dual space to the linear space of test functions

(59)

$$(\delta, \varphi) = \varphi(0)$$

$$(\delta', \varphi) = -\varphi'(0)$$

$$\delta(ax-b) = \frac{1}{|a|} \delta\left(x - \frac{b}{a}\right)$$

proof

$$\int \delta\left(\frac{ax-b}{y}\right) f(x) dx \quad dy = |a| dx$$

$$= \int \frac{dy}{|a|} \delta\left(\frac{y}{a}\right) f\left(\frac{y+b}{a}\right) = \frac{1}{|a|} f\left(\frac{b}{a}\right)$$

weak derivative

$$\int v(x) \varphi(x) dx = -\int u(x) \varphi'(x) dx$$

then $v(x)$ is the weak derivative of $u(x)$

$$\text{ex } \int \left[\frac{d}{dx} \text{sign } x \right] \varphi(x) = -\int \text{sign } x \varphi'(x)$$

$$= + \int_{-\infty}^{-\varepsilon} \varphi'(x) dx - \int_{\varepsilon}^{\infty} \varphi'(x) dx$$

$$= \varphi(-\varepsilon) + \varphi(\varepsilon) \rightarrow 2\varphi(0)$$

$$= \frac{d}{dx} \text{sign } x = 2\delta(x)$$

Example 2 wants derivative of $\log|x|$ (60)

$$\begin{aligned} I &= - \int_{-\infty}^{\infty} g'(x) \log|x| = \int_{-\infty}^{\infty} g(x) \frac{d}{dx} \log|x| \\ &= - \int_{\epsilon}^{\infty} g'(x) \log x - \int_{-\infty}^{-\epsilon} g'(x) \log(-x) \\ &= + \int_{\epsilon}^{\infty} g(x) \frac{1}{x} dx + \int_{-\infty}^{-\epsilon} g(x) \frac{1}{x} dx \\ &\quad + g(\epsilon) \log \epsilon - g(-\epsilon) \log \epsilon \\ &= \int_{-\infty}^{-\epsilon} \frac{g(x)}{x} dx + \int_{\epsilon}^{\infty} \frac{g(x)}{x} dx \\ &\quad + \underbrace{\log \epsilon (g(\epsilon) - g(-\epsilon))}_{\rightarrow 0} \end{aligned}$$

$$\equiv P\left(\frac{1}{x}\right) g$$

↑ principal value integral

only works if we choose the intervals symmetrically.