Green's functions

L diff-operator

$Ly = f$ solution is $y = L^{-1} f$

L^{-1} is called the Green's function

What is important for the uniqueness and existence of solutions is the Fredholm alternative which we discussed before for a finite vector space

1). Either $Ax = b$ has a unique solution
or $Ax = 0$ has a nontrivial solution
2) If $Ax = 0$ has n linearly independent solution then also $A^{+}x = 0$ does
3) If 2) holds then $Ax = b$ has no solutions unless b is $\perp$ to all solutions of $A^{+}x = 0$

This alternative is also valid for differential operator on a finite interval with L^{+} defined as discussed and the number of b.c. equal to the order of the differential equation

If the number of b.c. is different from the order $Ly = 0$ and $L^{+}y = 0$ generally will have a different # of solutions.

Example: $Ly = \frac{dy}{dx}$ $\quad y(0) = y(1) = 0$

then $L^{+} y = -\frac{dy}{dx}$ with no boundary conditions

$Ly=0$ only has $y = 0$ as solution

$\Rightarrow$ the solution of $Ly = f$ will be unique if it exists.

$L^{+} y = 0$ has solution $y = 1$ because there are no b c.

So there is no solution of $Ly = f$ unless $\langle 1, f\rangle = 0 = \int_0^1 f\, dx$

Indeed $\frac{d}{dx} y = f \Rightarrow y = \int_0^x f(x')dx'$

$y(1) = 0 \Rightarrow \int_0^1 f(x')dx' = 0 = \langle 1, f\rangle$

<u>Green's function</u>

$$Ly = f \Rightarrow y = L^{-1} f$$

$$(L^{-1})_{xy} \equiv G(x, y)$$

$L\, G = 1$ means $L_x G(x, y) = \delta(x-y)$

then $\quad y \equiv \int G(x,y) f(y)\, dy$

$$L_x y = \int L_x G(x,y) f(y)\, dy$$
$$= \int \delta(x-y) \; f(y) = f(x)$$

so y is a solution of the inhomogeneous equation.

What can we say about $G(x,y)$

- $G(x,y)$ must have a discontinuity at $x=y$. Otherwise it is not possiel to obtain a δ-function
- when $x \neq y$ then $L_x G(x,y) = 0$
- $G(x,y)$ must obey the boundary conditions of y at the ends of the interval

<u>Example</u> Green's function of Sturm Liouville equation

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\, y(x) = f(x)$$

on $[a,b]$ self adjoint b.c

$$\frac{y'(a)}{y(a)} = \tan\theta_L \qquad \frac{y'(b)}{y(b)} = \tan\theta_R$$

We want to solve

$$L_x \, G(x, \xi) = \delta(x - \xi)$$

with b.c of y.

$x \neq \xi$ then $G(x; \xi)$ solution of $L_x y = 0$

This equation has two independent solution y_L and y_R

$G(x, \xi)$ must be continuous at $x = \xi$ otherwise we would get the derivative of a δ-function

Its derivative must be discontinuous

So what may work is

$$G(x, \xi) = \begin{array}{ll} A \, y_L(x) \, y_R(\xi) & x < \xi \\ A \, y_R(x) \, L_L(\xi) & x > \xi \end{array}$$

then it solves $L_x \, G(x, \xi) = 0$ for $x \neq \xi$

but can we arrange things that we get a δ-function at $x = \xi$

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \left(\partial_{x'}(p\, G_{x'}(x',\xi)) + \underbrace{q(x')\, G(x',\xi)}_{\uparrow\ \mathcal{O}(\varepsilon)}\right) dx' = \underbrace{\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-x')\, dx'}_{1}$$

$$p(\varepsilon) \quad (-A\, y_L'(\xi)\, y_R(\xi) + A(y_L(\xi)\, y_R'(\xi)) = 1$$

$$= A\, p(\varepsilon)\, W = 1$$

$$W = \begin{vmatrix} y_L & y_R \\ y_L' & y_R' \end{vmatrix}$$

$$\Rightarrow \quad A = \frac{1}{p(\xi)\, W(\xi))}$$

$$\Rightarrow \quad G(x,\xi) = \frac{1}{p(\xi)\, W(\xi)}\, y_L(x)\, y_R(\xi), \quad x<\xi$$

$$= \frac{1}{p(\xi)\, W(\xi)}\, y_R(x)\, y_L(\xi), \quad x>\xi$$

We know that $W(x) = W(0)\, e^{-\int_0^x \frac{p_1}{p_0} dx'}$

$$\Rightarrow \frac{d}{dx}\, p\, W = p'\, W + p\, W(0)\, e^{-\int^x \frac{p_1}{p_0}} \times \frac{p_1}{p_0}(-1)$$

For Sturm Liouvill $p_1 = p_0' = p'$

$p = p_0$

$$\Rightarrow \frac{d}{dx}\, pW = 0 \qquad pW = \text{constant}$$

$$\Rightarrow \quad G(x,\xi) = G(\xi,x)$$

The general solution of the inhomogeneous equation is thus given by

$$y(x) = \int_a^b G(x,\xi)\, f(\xi)\, d\xi$$

$$= \int_a^x \frac{1}{pW}\, y_L(\xi)\, y_R(x)\, f(\xi)\, d\xi + \int_x^b \frac{1}{pW}\, y_L(x) y_R(\xi)\, f(\xi)\, d\xi$$

$$= \frac{1}{pW}\, y_R(x) \int_a^x y_L(\xi)\, f(\xi)\, d\xi + \frac{1}{pW}\, y_L(x) \int_x^b y_R(\xi)\, f(\xi)\, d\xi$$

<u>Example</u> $\quad -D_x^2\, y = f \qquad y(0) = y(1) = 0$

$$y_L = x \qquad y_R = 1 - x$$

$$\Rightarrow \quad G(x,\xi) = A x (1-\xi), \quad x < \xi$$
$$= A \xi (1-x), \quad x > \xi$$

$\Rightarrow$ solution

$$y(x) = \frac{A}{pW}(1-x) \int_0^x \xi\, f(\xi)\, d\xi + \frac{A}{pW}\, x \int_x^1 (1-\xi)\, f(\xi)\, d\xi$$

$$P_0 = -1 \qquad P_1 = 0 \quad \Rightarrow \quad W = W(0)$$

and $P_0 W$ = constant

A is determined by $A(-1)\left(\xi(-1) - 1(1-\xi)\right) = 1$

$$\Rightarrow A = 1$$

Initial value problems

$$Ly = \frac{dy}{dt} - Q(t)\,y = F(t) \qquad y(0) = 0$$

Green's function $\quad \frac{d}{dt} G(t,t') - Q(t) G(t,t') = \delta(t-t')$

$G(0,t') = 0$

$G(t,t') = 0$ for $t < t'$

Because $G(0,t') = 0$, then also $G(0+\delta t, t') = 0$ etc.

For $t > t'$ is in a solution of

$$\frac{d}{dt} y - Q(t)\,y = 0$$

$$\Rightarrow \quad y = y(0)\, e^{\int_0^t Q(t')\,dt'}$$

jump should be 1 to get $\delta(t-t')$

$\Rightarrow y(0) = 1$

and $G(t,t') = \theta(t-t')\, e^{\int_0^t Q(t'')\,dt''}$

solution of inhomogeneous equation

$$y(t) = \int_0^t F(t')\,\theta(t-t')\, e^{\int_0^t Q(t'')\,dt''}$$

Example: Forced h.o.

$$\ddot{x} + \Omega^2 x = F(t)$$

$$x(0) = x'(0) = 0$$

Green's function $\frac{d^2G(t,t')}{dt^2} + \Omega^2(t,t') = 0$

$$G(0,t') = 0 \qquad \partial_t G(0,t') = 0$$

$$\Rightarrow \quad G(t,t') = 0 \quad \text{for } t < t'$$

for $t > t'$ G is a solution of $\ddot{x} + \Omega^2 x = 0$ $\quad x = A\sin(\Omega t + \phi)$

$$\Rightarrow G(t;t') = \Theta(t-t') \, A \sin(\Omega t + \phi(t'))$$

with A and ϕ to be determined.

since we have a 2nd derivative $G(t,t')$ has to be continuous at $t' = t$, but its derivative has to change from zero to 1

$$\Rightarrow \quad A\sin(\Omega t + \phi(t'))\Big|_{t=t'} = 0 \Rightarrow \phi(t') = \Omega t'$$

$$\Omega A \cos(\Omega t + \phi(t'))\Big|_{t=t'} = 1 \Rightarrow \Omega A = 1$$

$$\Rightarrow G(t,t') = \frac{\Theta(t-t')}{\Omega} \sin \Omega(t-t')$$

General solution

$$x(t) = \int_0^t \frac{\theta(t-t')}{\Omega} \sin \Omega(t-t')\, F(t')\,dt'$$

Caldeira Leggett Model

Lagrangian

$$L = \frac{1}{2} \left(\dot{Q}^2 - (\Omega^2 - \Delta\Omega^2) Q^2 \right. - Q \sum_i f_i q_i + \sum_i \frac{1}{2}(\dot{q}_i^2 - \omega_i^2 q_i^2)$$

Q is macroscopic variable coupled to a heat bath of oscillators q_i

$$\Delta Q^2 = - \sum_i \frac{f_i^2}{\omega_i^2}$$

$$-\frac{\Omega^2}{2} Q^2 - Q \sum_i (f_i q_i) - \sum_i \frac{1}{2} \omega_i^2 q_i^2$$

$$= -\frac{\Omega^2}{2} Q^2 - \frac{1}{2} \sum_i \omega_i^2 \left(q_i + \frac{Q f_i}{\omega_i^2} \right)^2 + \sum_i \frac{1}{2} \frac{Q^2 f_i^2}{\omega_i^2}$$

this is cancelled by the ΔQ^2 term.

Eqs of motion

$$\ddot{Q} + (\Omega^2 - \Delta\Omega^2) Q + \sum f_i q_i = 0$$

$$\ddot{q}_i + \omega_i^2 q_i + f_i Q = 0$$

We now can solve for q_i using the Green's function for the ho

$$q_i = -\int_{-\infty}^{t} \frac{f_i}{\omega_i} \sin\omega_i(t-\tau)\, Q(\tau)\, d\tau$$

$$q_i(-\omega) = q_i'(-\omega) = 0$$

$$\Rightarrow \ddot{Q} + (\Omega^2 - \delta\Omega^2)Q + \sum_i -\int_{-\infty}^{t} \frac{f_i^2}{\omega_i} \sin\omega_i(t-\tau) \times Q(\tau)\, d\tau$$

only contribution for $\tau < t$ (memory function)

Introduce spectral function

$$J(\omega) \equiv \frac{\pi}{2} \sum_i \frac{f_i^2}{\omega_i} \delta(\omega - \omega_i)$$

$$\Rightarrow \sum_i (-1) \int_{-\infty}^{t} \frac{f_i^2}{\omega_i} \sin\omega_i(t-\tau)\, Q(\tau)\, d\tau$$

$$= -\frac{2}{\pi} \int_0^{\infty} d\omega \int_{-\infty}^{t} J(\omega) \sin\omega(t-\tau)\, Q(\tau)\, d\tau$$

$J(\omega)$ is the distribution of the heat bath frequencies

The coupling to these frequencies can become a friction force if $J(\omega) \propto \omega$

We choose $J(\omega) = \eta\omega \dfrac{\Lambda^2}{\Lambda^2+\omega^2}$

then $\dfrac{2}{\pi}\displaystyle\int_0^\infty d\omega\, J(\omega) \sin\omega t$ $(t-\tau)$

$$= \frac{2}{\pi}\int_0^\infty d\omega \frac{\eta\omega\Lambda^2}{\Lambda^2+\omega^2}\sin\omega t$$

$\sin\omega t = \dfrac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$\omega \to -\omega$

$\displaystyle\int_0^\infty \to \int_{-\infty}^0$

$$= \frac{1}{\pi i}\int_{-\infty}^\infty \frac{\eta\omega\Lambda^2}{\Lambda^2+\omega^2} e^{i\omega t}$$

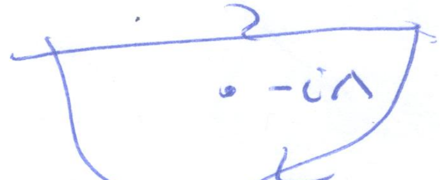
$t > 0$

$\dfrac{\omega}{\Lambda^2+\omega^2} = \dfrac{1}{2}\left(\dfrac{1}{\omega+i\Lambda}+\dfrac{1}{\omega-i\Lambda}\right)$

$= \dfrac{2\pi i}{\pi i}\eta\Lambda^2 e^{-\Lambda t}$ $t > 0$

$t < 0$

$= \dfrac{2\pi i}{2\pi i}(-1)\,\eta\Lambda^2 e^{\Lambda t}$ $t < 0$



$= \eta\Lambda^2 \,\mathrm{sgn}(t)\, e^{-\Lambda|t|}$

(-1) sign for counter clockwise contours

So the $J(\omega)$ term becomes

$$-\int_{-\infty}^{t} \eta\Lambda^2 e^{-\Lambda|(t-\tau)|} Q(\tau)\,\mathrm{sgn}(t-\tau)\,d\tau$$

$\tau < t \Rightarrow \mathrm{sgn}(t-\tau) = 1$

$Q(\tau) = Q(t) + (\tau - t)\dot{Q}(t) + \frac{1}{2}(t-\tau)^2\ddot{Q}(t)$

$$\Rightarrow -\int_{-\infty}^{t} \eta \Lambda^2 e^{-\Lambda(t-\tau)} \left(Q(t) + (\tau - t)\dot{Q}(t) + \tfrac{1}{2}(\tau-t)^2 \ddot{Q}(t) \right)$$

$$= -\eta \Lambda Q(t) + \eta \dot{Q}(t) - \frac{\eta}{2\Lambda} \dddot{Q}(t) + \dots$$

$$-\Delta \Omega^2 = \sum_i \frac{f_i}{\omega_i^2} = \frac{2}{\pi}\int_0^\infty \frac{\gamma(\omega)}{\omega}\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\eta \Lambda^2}{\Lambda^2 + \omega^2}\, d\omega = \eta \Lambda$$

$\Rightarrow$ $-\eta \Lambda Q(t)$ term is cancelled

Taking the limit $\Lambda \to \infty$
gives

$$\ddot{Q} + \eta \dot{Q} + \Omega^2 Q = 0$$

$\uparrow$ friction term

Example with $Ly = 0$
and $f \perp \{y \mid L^+ y = 0\}$

$Ly = -\partial_x^2 y = f(x)$ $\quad y'(0) = y'(1) = 0$

$Ly = 0$ has nontrivial solution $y = 1$

$L^+ = L$ $\quad Ly = f$ has solutions if
$f \perp 1$ i.e. $\int_0^1 f\,dx = 0$.

Green's function $-\partial_x^2 G(x, \xi) = \delta(x - \xi)$
is not possible
because $y = 1$
is only nontrivial solution satisfying $y'(0) = 0$
or $y'(1) = 0$

Instead we look at $-\partial_x^2 G(x, \xi) = \delta(x-\xi) - 1$
then both sides integrate to zero

$x \neq \xi$ then $-\partial_x^2 y = -1 \Rightarrow y = \frac{1}{2}x^2 + A + Bx$

$y'(0) = B = 0 \quad y'(1) = 1 + B = 0$

$\Rightarrow y_L = A + \frac{1}{2}x^2 \qquad y_R = A' - x + \frac{1}{2}x^2$

Greens Function should be continuous
at $x = \xi$

Because we have an inhomogeneous equation for the Green's function we cannot use the usual trick of superimposing solution i.e. $y_L(x)\, y_R(\xi)$ for $x<\xi$ and $y_R(x)\, y_L(\xi)$ for $x>\xi$

We use that: $G(x,\xi) = A + \frac{1}{2}x^2 \quad x<\xi$

$$= A' - x + \tfrac{1}{2}x^2 \quad x>\xi$$

A, A' depend on ξ

G is continuous at $x=\xi \Rightarrow$

$$A + \tfrac{1}{2}\xi^2 = A' - \xi + \tfrac{1}{2}\xi^2$$

$$\Rightarrow \quad A' = A + \xi$$

at $x=\xi$ the derivative should jump by -1

$x = \xi - \varepsilon \qquad \partial_x G(x,\xi) = \xi$

$x = \xi + \varepsilon \qquad \partial_x G(x,\xi) = -1 + \xi$

OK

$$\Rightarrow \quad G(x,\xi) = \begin{cases} A + \frac{1}{2}x^2 & x<\xi \\ A + \xi - x + \frac{1}{2}x^2 & x>\xi \end{cases}$$

A is not determined and we can chose it such that $G(x,\xi) = G(\xi,x)$

$$\Rightarrow \qquad A = \tfrac{1}{2}\xi^2 - \xi$$

$$\Rightarrow \quad G(x,\xi) = -\xi + \tfrac{1}{2}(x^2+\xi^2) \quad x<\xi$$
$$\qquad\qquad\quad -x + \tfrac{1}{2}(x^2+\xi^2) \quad x>\xi$$

General solution of inhomogeneous equation is

$$g(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi \quad + \text{constant}$$

Symmetrizing the Green's function

$L_x G(x,\xi) = \delta(x-\xi)$ bc B

$L_x^+ G^+(x,\xi') = \delta(x-\xi')$ bc B^+

Lagrange identity

$$w\left(u^* L v - v\left(L^+ u\right)^*\right) = \frac{d}{dx} Q(u,v)$$

integrate $w=1$

$$Q(G^+,G)\Big|_a^b = \int_a^b dx\left[G_{(x,\xi)}\left(L^+ G^+(x,\xi')\right)^* - G^{+*}_{(x,\xi')}\left(L\, G(x,\xi)\right)\right]$$

$$= \int_a^b dx\left(G_{(x,\xi)}\,\delta(x-\xi') - (G^+)^*_{(x,\xi')}\,\delta(x-\xi)\right)$$

$$= G(\xi',\xi) - G^{+*}(\xi,\xi')$$

For self adjoint b.c. $Q(G^+,G)\Big|_a^b = 0$

$\Rightarrow$ $G(x,\xi)$ is Hermitian

<u>Example</u>: $L = \frac{d}{dx}$ $y(0)=0$

$L^+ = -\frac{d}{dx}$ $y(1)=0$

$G(x,\xi) = \Theta(x-\xi)$ ~~ξ~~ $G(x,\xi) = G^+(\xi,x)$

$G^+(x,\xi) = \Theta(\xi-x)$

Eigenfunction expansion of Green's function

$L^+ = L \qquad L\varphi_n = \lambda_n \varphi_n$

Then $\quad LG = GL = 1$

$$G(x,y) = \langle x | L^{-1} | y \rangle = \sum_{nm} \langle x | n \rangle \langle n | L^{-1} | m \rangle \langle m | y \rangle$$
$$= \sum_{nm} \frac{\varphi_n(x)\, \delta_{nm}\, \varphi_m^*(y)}{\lambda_n}$$
$$= \sum_n \frac{\varphi_n(x)\varphi_n^*(y)}{\lambda_n}$$

example $\quad L = -\partial_x^2 \qquad y(0) = y(1) = 1$

$\varphi_n = \sqrt{2} \sin n\pi x \qquad \int_0^1 \varphi_n^2(x)\, dx = 1$

$\lambda_n = n^2\pi^2$

then $\quad G(x,y) = \sum \frac{2}{n^2\pi^2} \sin n\pi x \sin n\pi y$

$$= \begin{cases} x(1-y) & x < y \\ y(1-x) & x > y \end{cases}$$

$$\partial_y G(x,y) = \sum_{n=1}^{\infty} \sin n\pi x \cos n\pi y \frac{2}{\pi n}$$

$x < y \qquad -x$

$x > y \qquad \frac{i}{2\pi}\left(2\pi i \cdot i\pi(x-y) \quad \pi i(x+y)\right) = 1 - x$

$= \frac{i}{2\pi} \log e^{i\pi(x-y)} e^{i\pi(x+y)}$

(117)

$$\frac{2i}{4\pi}\left[\log\left(\frac{1-e^{+i\pi(x-y)}}{1-e^{-\pi i(x-y)}}\right)+\log\frac{(1-e^{i\pi(x+y)})}{1-e^{-i\pi i(x+y)}}\right.$$

$$=\frac{i}{2\pi}\log(-)e^{+i\pi(x-y)} \quad +\frac{i}{2\pi}\log(-1)e^{i\pi(x+y)}$$

$x<y \qquad \frac{i}{2\pi}\left(\pi i+\pi i(x-y)\right) \qquad i\pi(x+y)-\pi i)$

$$=\frac{i}{2\pi}\, i x 2\pi = -x$$

$x>y \qquad \frac{i}{2\pi}\left(i\pi(x-y)-\pi i \qquad i\pi(x+y)-\pi i\right)$

$$=\frac{i}{2\pi}\left(2\pi i x-2\pi i\right)=1-x$$

Causality and analyticity

$$G(t-\tau)=0 \qquad t<\tau$$

causal property

Fourier transform $\quad$ c $G(t<0)=0$

$$\tilde{G}(\omega)=\int_0^\infty e^{i\omega t}\,G(t)\,dt$$

integral converges if $\operatorname{Im}(\omega)>0$

$\Rightarrow$ $\tilde{G}(\omega)$ is analytic in upper half plane

Example Damped ho

$$G(t) = \frac{\theta(t)}{\Omega} e^{-\gamma t} \sin \Omega t$$

$$\tilde{G}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \frac{\theta(t)}{\Omega} e^{-\gamma t} \sin \Omega t$$

$$= \frac{1}{\Omega}\int_{0}^{\infty} e^{i\omega t} e^{-\gamma t} \sin \Omega t$$

$$= \frac{1}{2i}\frac{1}{\Omega}\int_{0}^{\infty} e^{i\omega t} e^{-\gamma t} \left(e^{i\Omega t} - e^{-i\Omega t}\right)$$

$$= \frac{1}{2i}\frac{1}{\Omega}\int_{0}^{\infty} \left[e^{i(\omega+\Omega)t - \gamma t} - e^{i(\omega-\Omega)t - \gamma t}\right] dt$$

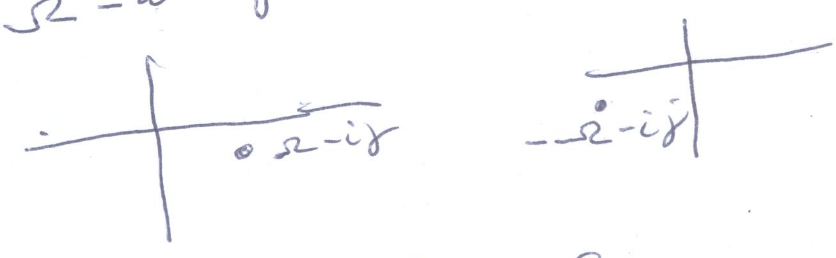
integral converges $\gamma > 0$

$$= \frac{1}{\Omega}\frac{1}{2i}\left(\frac{1}{\underset{\uparrow\,\text{minus}}{-}i(\omega+\Omega)+\gamma} - \frac{1}{-i(\omega-\Omega)+\gamma}\right) = \frac{1}{2i}\frac{2i\Omega}{\Omega\left(\Omega^2 - (\omega+i\gamma)^2\right)}$$

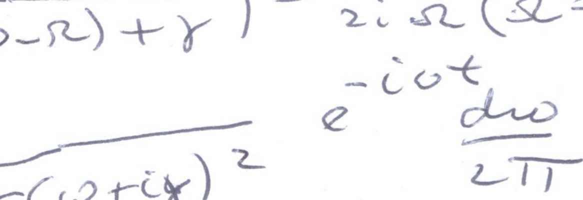
$$G(t) = \int_{-\infty}^{\infty} \cancel{\omega\Omega}\, \frac{1}{\Omega^2 - (\omega + i\gamma)^2} e^{-i\omega t} \frac{d\omega}{2\pi}$$

$$\parallel$$

$$\frac{1}{2\Omega}\left(\frac{1}{\Omega - \omega - i\gamma} + \frac{1}{+(\Omega+\omega)+i\gamma}\right)$$



pole is in negative complex plane.



$t > 0$ close contour

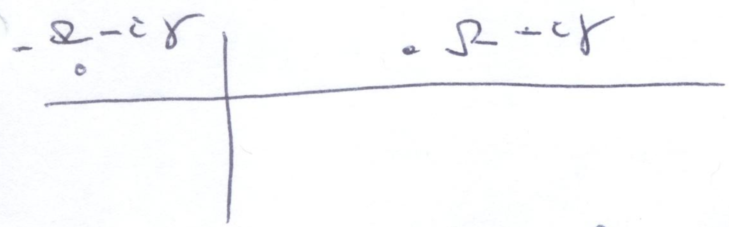
$t < 0$ close contour $\Rightarrow$ integral $= 0$

$$\theta(t)\frac{2\pi i}{4\pi\Omega}\left(e^{-i(\Omega - i\gamma)t} - e^{-i(-\Omega - i\gamma)t}\right)$$

$$= \theta(t)\frac{1}{\Omega} e^{-\gamma t} \sin \Omega t$$

minus sign for clockwise contour

What happens if $\gamma < 0$



poles are in upper half plane

then integral should vanishes if $t > 0$

$$\Rightarrow \theta(-t)\,\frac{2\pi i}{4\pi\Omega}(-1)\left(e^{-i(\Omega - i\gamma)t} - e^{-i(-\Omega - i\gamma t)}\right)$$

$$= -\frac{\theta(-t)}{\Omega}\, e^{|\gamma| t}\, \sin \Omega t$$

Plemelj Formula

Fourier transform of the h o is of the form $\dfrac{1}{\omega' - \omega}$

when we have many oscillators it becomes

$$\frac{1}{\pi}\int_a^b \frac{\rho(\omega')}{\omega' - \omega}\, d\omega'$$

to make the integral convergent if $\omega' \in [a, b)$ then $\omega \to \omega \pm i\varepsilon$

Then

$$\frac{1}{2}\left(f(\omega+i\varepsilon)\right) + -\frac{1}{2} f(\omega - i\varepsilon)$$

$$= \frac{1}{2\pi}\int_a^b \frac{\rho(\omega')}{\omega'-\omega-i\varepsilon} - \frac{\rho(\omega')}{\omega'-\omega+i\varepsilon}$$

$$= \frac{1}{2\pi}\int_a^b \rho(\omega') \underbrace{\frac{2i\varepsilon}{(\omega'-\omega)^2+\varepsilon^2}}_{\varepsilon\to 0 \quad 2i\pi\,\delta(\omega'-\omega)}$$

$$= i\,\rho(\omega)$$

$\Rightarrow f$ is discontinuous across real axis

$$\frac{1}{2}\left(f(\omega+i\varepsilon) + f(\omega-i\varepsilon)\right)$$

$$= \frac{1}{2\pi}\int_a^b \rho(\omega')\frac{2(\omega'-\omega)}{(\omega'-\omega)^2+\varepsilon^2}$$

$$\varepsilon \to 0$$

$$= \frac{1}{\pi}\, P\int_a^b \frac{\rho(\omega')}{\omega'-\omega}$$

These formula are called the Plemelj formulae

$$\Rightarrow \rho(\omega) = \operatorname{Im} f(\omega \quad)$$

$$\operatorname{Re} f(\omega \quad) = \frac{1}{\pi} P\int \frac{\operatorname{Im} f(\omega' \quad)}{\omega'-\omega}\, d\omega'$$

Resolvent

$$R_\lambda = (L - \lambda \mathbb{1})^{-1}$$

in terms of eigenfunctions

$$R(x, \xi) = \sum_{n,m} \int \varphi_n^*(\xi)\, (L - \lambda \mathbb{1})^{-1} \varphi_m(x)$$

$$= \sum_n \frac{\varphi_n^*(\xi)\ \varphi_n(x)}{\lambda_n - \lambda}$$

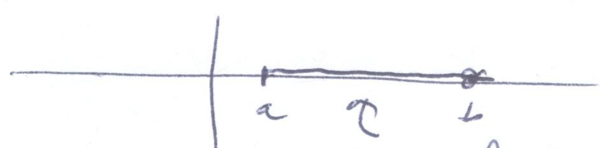
$L\, \varphi_n$ $\quad L\, \varphi_n = \lambda_n \varphi_n$

if the spectrum of L is continuous

$$\sum_n \to \int d\mu\, \rho(\mu)$$

density of states

The resolvent is discontinuous over the continuous part of the spectrum



branch cut $\sigma(\mu) = [a, b]$

$$\mathrm{Tr}\, R_\lambda = \int dx\ R_\lambda(x, x)$$

$$= \sum_n \frac{1}{\lambda_n - \lambda}$$

continuous spectrum $\quad \displaystyle\int \frac{\rho(\lambda')}{\lambda' - \lambda}\, d\lambda$

Plemelj formula

$$\operatorname{Im} \lim_{\varepsilon \to 0} \operatorname{Tr} R_{\lambda + i\varepsilon} = \rho(\lambda)$$

$$\rho \in \mathbb{R}$$

<u>Example</u> $L = -i\partial_x$

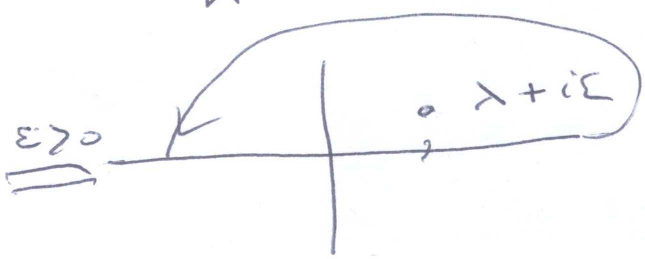
$$\mathcal{D}(L) = \{y,\ Ly \in L^2(\mathbb{R})\}$$

$$-i\partial_x \, e^{ikx} = k\, e^{ikx}$$

spectrum is the real line.

$$(-i\partial_x - \lambda)^{-1} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ik(x-\xi)}}{k-\lambda}\, dk$$

$\lambda \to \lambda + i\varepsilon$ $\qquad \underline{\underline{\varepsilon > 0}}$



$x - \xi > 0$ can close contours in upper half plane $\quad \dfrac{2\pi i}{2\pi}\, e^{i(\lambda + i\varepsilon)(x-\xi)}$

$x - \xi < 0$ can close contour in lower half plane $\Rightarrow$ integral vanishes

$\underline{\underline{\varepsilon > 0}}$ $\quad (-i\partial_x - \overset{-i\varepsilon}{\lambda}) = i\,\theta(x-\xi)\, e^{i(\lambda+i\varepsilon)(x-\varepsilon)}$

When $\underline{\underline{\varepsilon < 0}}$ the contour integral is nonvanishing for the lower contour.

– sign for clockwise contour

$$(-i\partial_x - \overset{i\varepsilon}{\lambda}) = -i\,\theta(\xi - x)\, e^{i(\lambda+i\varepsilon)(x-\xi)}$$

$$\mathrm{Tr}\,(-i\partial_x - \lambda - i\varepsilon) = \frac{1}{2\pi}\int dx \, \frac{e^{ik(x-x)}}{k - \lambda - i\varepsilon}\, dk$$

$$\downarrow$$

$$L$$

$$= \frac{L}{2\pi} \int dk \, \frac{k - \lambda + i\varepsilon}{(k-\lambda)^2 + \varepsilon^2}$$

take principle value for real part
this gives zero

$\varepsilon > 0$

Imaginary part $\frac{L}{2\pi} \underbrace{\int \frac{dk \; i\varepsilon}{(k-\lambda)^2 + \varepsilon^2}}_{\pi i \,\mathrm{sign}(\varepsilon)}$

$$= \frac{L}{2\pi} \pi i = \frac{L i}{2} \;\mathrm{sgn}\; \varepsilon$$

$$\int_{-\infty}^{\infty} \frac{dk \; \varepsilon}{(k-\lambda)^2 + \varepsilon^2} = \int_{-\infty}^{\infty} \frac{dk \; \varepsilon}{k^2 + \varepsilon^2}$$

$$= \int_{-\infty}^{\infty} \frac{dk \; \varepsilon}{k^2 |\varepsilon| \,(k^2 + 1)} = \frac{\varepsilon}{|\varepsilon|}\, \pi$$

The discontinuity of the resolvant
gives the spectral density

Gelfand – Dikii equations

- Green's Functions are typically local $G(x,y)$ only depend on the coefficients of the differential eq near x and y.

This can be shown explicitly for $G(x,x)$ of the Schrödinger eq.

$$(-\partial_x^2 + q(x) + \lambda)\psi = 0$$

$$G(x,\xi) \sim u(x)\, v(\xi)$$

with u and v independent solution.

Let $D(x) = u(x)\, v(x)$

then $\partial_x^2 D(x) = (\partial^2 u)\, v + (\partial^2 v)\, u + 2\partial_x u\, \partial_x v$

$$= v\,(q(x)+\lambda)u + u\,(q(x)+\lambda)v + 2\partial_x u\, \partial_x v$$

$$\partial_x D(x) = (\partial_x u)\, v + (\partial_x v)\, u$$

$$\partial_x^3 D(x) = (q(x)+\lambda) \cdot 2\partial_x D + (\partial_x q) 2uv$$
$$+ 2\partial_x^2 u\, \partial_x v + 2\partial_x^2 v\, \partial_x u$$

$$= (q(x)+\lambda)\, 2\partial_x D + (\partial_x q) 2uv$$
$$+ 2\partial_x v\,(q(x)+\lambda)u + 2\partial_x u\,(q(x)+\lambda)v$$

$$= (q(x)+\lambda)\, 4\, \partial_x D + 2D\, \partial_x q$$

$$\Rightarrow \partial_x^3 D - 4 q\, \partial_x D - 2D\, \partial_x q = 4\lambda\, \partial_x D$$

$$\Rightarrow (q\partial_x + \partial_x q - \tfrac{1}{2}\partial_x^3) D = -2\lambda \partial_x D$$

note that

Gelfand-Dikii equ.

$$\partial_x q D = (\partial_x q) D + q \partial_x D$$

For $q = 0$ we have the

$$G(x,y) = \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(|x-y|)}$$

<u>proof</u>:

$$\partial_x G(x,y) = \frac{1}{2\sqrt{\lambda}}(-\sqrt{\lambda})\,\mathrm{sign}(x-y) \times e^{-\sqrt{\lambda}|x-y|}$$

$$\partial_x^2 G(x,y) = \frac{1}{2\sqrt{\lambda}} \lambda\, e^{-\sqrt{\lambda}|x-y|} - \frac{1}{2} 2\,\delta(x-y)\, e^{-\sqrt{\lambda}|x-y|}$$

$$\Rightarrow \partial_x^2 G(x,y) = \lambda\, G(x,y) - \delta(x-y)$$

$$\Rightarrow (-\partial_x^2 + \lambda)\, G(x,y) = \delta(x-y)$$

$$\Rightarrow D(x,x) = \frac{1}{2\sqrt{\lambda}}$$

Then we can solve the Gelfand-Dikii perturbative in q.