1. Functional derivatives

Functional $\qquad \mathcal{F} : C^\infty(\mathbb{R}) \to \mathbb{R}$

smooth functions $\mathbb{R} \to \mathbb{R}$ $\qquad$ real number

derivative $\qquad f(x+\delta x) = f(x) + \delta x\, f'(x) + \ldots$

Taylor.

For functional derivative we do the same

$$\mathcal{F}(f+\delta f) = \mathcal{F}(f) + \underbrace{\frac{\delta \mathcal{F}}{\delta f}}_{\text{functional derivative}} \cdot \delta f$$

<u>Example</u> $\qquad \mathcal{F}(f) = \int_{-\infty}^{\infty} f^2(x)\, dx$

$$\mathcal{F}(f+\delta f) = \int_{-\infty}^{\infty} (f(x) + \delta f(x))^2$$

$$= \mathcal{F}(f) + 2\int_{-\infty}^{\infty} f(x)\,\delta f(x) + \mathcal{O}(\delta f(x))^2$$

$$\Rightarrow \quad \frac{\delta \mathcal{F}}{\delta f} = 2 f(x)$$

We see that the $\cdot$ in $\frac{\delta \mathcal{F}}{\delta f} \cdot \delta f$ means integration

Example 2

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} (f'(x))^2 \, dx$$

$$\mathcal{F}(f+\delta f) = \int_{-\infty}^{\infty} (f'(x))^2 + 2 f'(x) \, \delta f'(x) + \cdots)$$

$$= \mathcal{F}(f) + \int 2 f'(x) \, \delta f'(x) \, dx$$

$$\qquad\qquad\qquad \shortparallel$$

$$- \int 2 f''(x) \, \delta f(x)$$

partial integrat

$$\Rightarrow \quad \frac{\delta \mathcal{F}}{\delta f} = -2 f''(x)$$

General case $\qquad \mathcal{F} = \int_{-\infty}^{\infty} L(f, f') \, dx$

depends on

f and f'

eg $L = f^2 + f'^2$

$$\mathcal{F}(f+\delta f) = \int_{-\infty}^{\infty} L(f, f') + \frac{\partial L}{\partial f} \delta f + \frac{\partial L}{\partial f'} \delta f'$$

$$= \mathcal{F}(f) + \int \left(\frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'} \right) \delta f$$

partial integrate

$$\Rightarrow \frac{\delta \mathcal{F}}{\delta L} = \frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'}$$

at the endpoints we have $\delta f(x_1) = \delta f(x_2) = 0$. Here $x_1 = -\infty$ and $x_2 = \infty$ but generally they can be anything.

Application: Finding the extrema of a functional $\frac{\delta \mathcal{F}}{\delta f} = 0$.

Euler Lagrange equations

$$S = \int_{t_1}^{t_2} L(x(t), x'(t))\, dt$$

stationary points of action

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x(t)} - \frac{d}{dt}\frac{\partial L}{\partial x'(t)} = 0$$

example: $L = \frac{1}{2}\dot{x}^2 - V(x)$

$$\frac{\partial L}{\partial x} = -V'(x) \qquad \frac{\partial L}{\partial x'} = \dot{x}$$

$$\Rightarrow -V'(x) - \frac{d}{dt}\dot{x} = 0$$

$$\Rightarrow \ddot{x} = -V'(x)$$ Newton Equations

First integral

if L does not have an explicit time dependence, we have a first integral.

$$\frac{\partial L}{\partial t} = 0 \Rightarrow L(x(t), x'(t), t) = L(x(t), x'(t))$$

Then $\frac{d}{dt}\left(L - x'\frac{\partial L}{\partial x'}\right)$

$$= \frac{\partial L}{\partial x} x' + \cancel{\frac{\partial L}{\partial x'} x''} - \cancel{x''\frac{\partial L}{\partial x'}}$$

$$- x'\frac{\partial}{\partial x}\frac{\partial L}{\partial x'} x' - x'\frac{\partial^2 L}{\partial x'^2} x''$$

$$= x'\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x'}\right) = 0$$

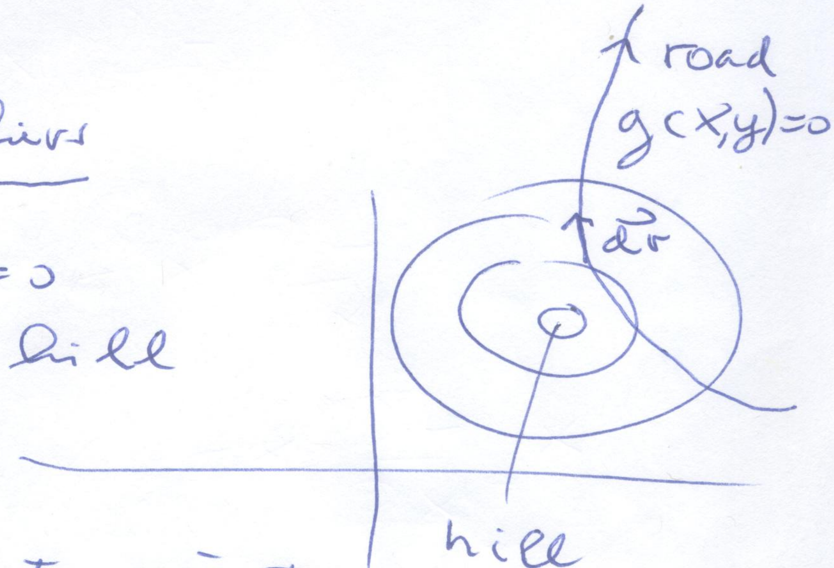
by the EL eqs.

$L - x'\frac{\partial L}{\partial x'}$ is called the first integral

What is the first integral for $L = \frac{1}{2}\dot{x}^2 - V(x)$

$$\frac{1}{2}\dot{x}^2 - V(x) - \dot{x}\,\dot{x} = -\frac{1}{2}\dot{x}^2 - V(x)$$

this is minus the energy and we know that it is conserved.



Lagrange multipliers

road $g(x,y)=0$
contour map of hill
$f(x,y) = h$.

what is the highest point on the road.

At the highest point we have

$$df\Big|_{\text{along road}} = 0 \qquad df = \underbrace{\vec{\nabla} f}_{(\partial_x f,\, \partial_y f)} \cdot \vec{dr} \ \text{along road}$$

$$dg = 0 \Rightarrow \vec{\nabla} g \cdot \vec{dr} = 0$$

$\Rightarrow$ when $df = 0$ then $\vec{\nabla} g$ and $\vec{\nabla} f$ are parallel.

$\Rightarrow \quad \vec{\nabla} f = +\lambda \vec{\nabla} g$ 2 eqs

$g(x,y) = 0$ 1 eq

λ is called a Lagrange multiplier.

- 3 eqs for 3 unknowns (x, y, λ)
- can be solved.

Example $x^2 + y^2 = h$

$g(x,y) = ax + by - c$

$\vec{\nabla} f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$\vec{\nabla} g = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$ $ax + by = c$

$\Rightarrow \quad 2x = \lambda a$

$2y = \lambda b$

$\Rightarrow \quad a \frac{\lambda a}{2} + b \frac{\lambda b}{2} = c$

$\Rightarrow \lambda = \frac{c}{\frac{1}{2}(a^2+b^2)}$ $x = \frac{ca}{a^2+b^2}$

$y = \frac{bc}{a^2+b^2}$

Lagrange multipliers with many constraint

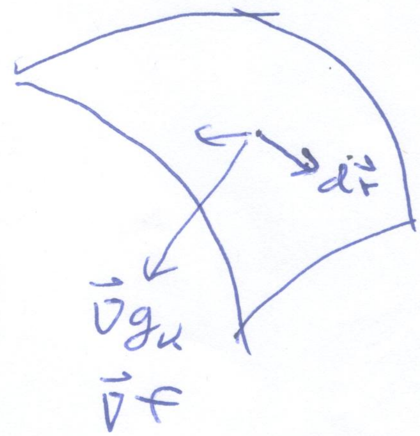
Constraints $\quad g_1 = \cdots = g_K = 0$

then $\quad dg_K = 0 = \vec{\nabla} g_K \cdot d\vec{r}$

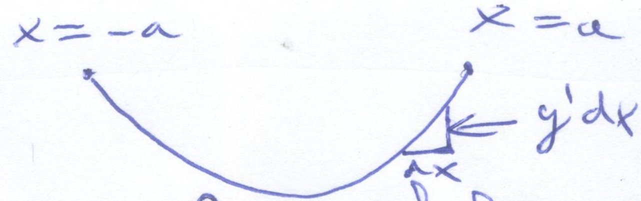
$$df = \vec{\nabla} f \cdot d\vec{r} = 0 \quad \text{(on constraint)}$$

$$\Rightarrow \quad \vec{\nabla} f = \sum_{K=1}^{\infty} \lambda_K \vec{\nabla} g_K$$

$d\vec{r}$ is tangent to constraint surface



The catenary



chain of fixed length between two points.

Its shape is called the catenary.

$$\text{Lenght} = \int_{-a}^{a} \sqrt{1+y'^2}\, dx$$

$$\text{Energy} = \int_{-a}^{a} \underset{\text{mass density}}{\rho} \sqrt{1+y'^2}\, y\, dx$$

we have to minimize the energy subject to the constraint that

$$L = \int_{-a}^{a} \sqrt{1+y'^2}\, dx$$

So we have to find the stationary point of

$$\int_{-a}^{a} \rho \sqrt{1+y'^2}\, y\, dx - \lambda \int_{-a}^{a} \sqrt{1+y'^2}\, dx$$

This is a function of y and y'.

EL equation.

$$\rho \sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{\rho y}{\sqrt{1+y'^2}}\, y' - \frac{\lambda y'}{\sqrt{1+y'^2}} \right) = 0$$

First integral

$$L - y' \frac{\partial L}{\partial y'} = \text{constant}$$

$$\Rightarrow \rho \sqrt{1+y'^2}\, y - \lambda \sqrt{1+y'^2} - y' \left(\frac{\rho y y' - \lambda y'}{\sqrt{1+y'^2}} \right)$$

$$= \text{constant}$$

$$\Rightarrow (\rho y - \lambda)(1 + \cancel{y'^2}) - \cancel{y'^2 (\rho y - \lambda)} = c\sqrt{1+y'^2}$$

$$\Rightarrow \quad \rho y - \lambda = c \sqrt{1+y'^2}$$

put $\rho = 1$ $\Rightarrow$ $y - \lambda = c\sqrt{1+y'^2}$

It is clear that the solution is of the form $y = \lambda + b_1 \cosh b_2 x$

substitute

$$b_1 \cosh b_2 x = c\sqrt{1 + b_1^2 b_2^2 \sinh^2 b_2 x}$$

$$\Rightarrow b_1 b_2 = \pm 1$$

$$b_1 \cosh b_2 x = c \cosh b_2 x$$

$$\Rightarrow b_1 = c \quad \Rightarrow y = \lambda + c \cosh \frac{x}{c} ,$$

we still have to impose the constraint that the length is L

$$\int_{-a}^{a} \sqrt{1+y'^2}\, dx = L.$$

$$= \int_{-a}^{a} \sqrt{1+\sinh^2\frac{x}{c}}\, dx = \int_{-a}^{a} \cosh\frac{x}{c}\, dx$$

$$\Rightarrow \quad 2c \sinh\frac{a}{c} = L.$$

We can put $\lambda = 0$ because that is just a shift of the y-axis

if we fix the y axis by $y(\pm a) = 0$

then $\quad \lambda + c\cosh\frac{a}{c} = 0$

$$\Rightarrow \quad \lambda^2 = c^2\left(1+\frac{L^2}{4c^2}\right)$$

$$\Rightarrow \quad \frac{\lambda^2}{c^2} - \frac{L^2}{4c^2} = 1$$

$$\Rightarrow \quad \lambda = \pm\sqrt{c^2+\frac{L^2}{4}}$$

$$\Rightarrow \quad y = -\sqrt{c^2+\frac{L^2}{4}} + c\cosh\frac{x}{c}$$

$$2c \sinh\frac{a}{c} = 1$$

Action principle

action $S = \int_{t_1}^{t_2} L \, dt$

$\uparrow$ Lagrangian.

classical equations of motion are the stationary points of S

$\delta S = 0 \qquad L = L(q, \dot q, t)$

Euler-Lagrange eqs $\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0$

Example central force problem

$L = T - V = \frac{1}{2} m(\dot r^2 + r^2 \dot\theta^2) - V(r)$

EL eqs : r ; $m \ddot r - m r \dot\theta^2 - V'(r) = 0$

θ ; $\frac{d}{dt}(m r^2 \dot\theta) = 0$

angular momentum conservation.

first integral

$E = r \frac{\partial L}{\partial \dot r} + \theta \frac{\partial L}{\partial \dot\theta} - L$

$= \frac{1}{2} m(\dot r^2 + r^2 \dot\theta^2) + V(r)$

$\frac{dE}{dt} = 0$

Noether's theorem

symmetry $\Longleftrightarrow$ conservation law
time translation — energy
rotational invariant — angular momentum

Noether's theorem makes the relation between symmetry and conservation law explicit.

Let us look at

$$S = \int_0^T \left(\tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2) - V(r)\right) dt$$

integrand is invariant for $\theta \to \theta + \varepsilon\alpha$

function of t — constant

At the extremum $\delta S = 0$ the action is stationary for all variations even time dependent variations. So we look at

$$\theta(t) \to \theta(t) + \varepsilon(t)\alpha$$

$$\delta S = \int_0^T dt\, \tfrac{1}{2}mr^2(\dot\theta + \dot\varepsilon\alpha)^2 - \int_0^T \tfrac{1}{2}mr^2\dot\theta^2$$

$$= \int_0^T dt\, \tfrac{1}{2}mr^2\, 2\,\dot\theta\,\dot\varepsilon\,\alpha$$

choose boundary condition $\varepsilon(0) = \varepsilon(T) = 0$ and partial integrate

$$\delta S = -\int_0^T dt\, \alpha \frac{d}{dt}\left(\tfrac{1}{2}mr^2\dot\theta\right)\varepsilon = 0$$

for all $\varepsilon(t)$

$$\Rightarrow \frac{d}{dt}\left(\tfrac{1}{2}mr^2\dot\theta\right) = 0$$

angular momentum conservation

Many degrees of freedom

The action principle is also valid for field theories

field $\varphi(x_\mu)$

action $S(\varphi) = \int L\,dt = \int \underbrace{\mathcal{L}(x^\mu, \varphi, \partial_\mu \varphi)}_{\text{Lagrange density}} d^{d+1}x$

$$\delta S = \int \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \partial_\mu \varphi) \, d^{d+1}x$$

$$= \int \frac{\delta \mathcal{L}}{\partial \varphi} \delta\varphi - \frac{d}{dx_\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta\varphi) \, d^{d+1}x$$

$\delta\varphi$ vanishes on boundaries

Euler Lagrange equation.

$$\frac{\delta \mathcal{L}}{\delta \varphi} - \frac{d}{dx_\mu} \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} = 0$$

Example: vibrating string $L = \int_0^T dx \left(\frac{1}{2}\rho \dot{y}^2 - \frac{T}{2} y'^2\right)$

$y' = \frac{dy}{dx}$

$S = \int L\,dt$

EL $\quad -\frac{d}{dt} \rho \dot{y} + \frac{1}{2} \frac{d}{dx} \frac{dy}{dx} = 0$

$\Rightarrow \rho \ddot{y} - T y'' = 0.$