Lecture #3

- catenary
  used first integral of EL
- field theory

Today:
- action principle
- Noether's theorem
- review of linear algebra

Vector space V

$x + y \qquad \lambda x \qquad x \in V$

$\uparrow \qquad y \in V$

scalar

$x+y = y+x \qquad (x+y)+z = x + (y+z)$

$0 + x = x \qquad -x + x = 0$

$\lambda(x+y) = \lambda x + \lambda y$

$(\mu+\lambda) x = \mu x + \lambda x$

$(\lambda \mu) x = \lambda (\mu x)$

$1 x = x$

finite vector space

n dimensional: $\exists$ n basis elements

e_k such that $\lambda_1 e_1 + \dots + \lambda_n e_n = 0$

implies $\lambda_1 = \dots = \lambda_n = 0$

any element of V can be expressed in this basis

$$x = x_1 e_1 + \dots + x_n e_n$$

if e_k and e_k' are basis then

$$e_k' = a_{k\ell} e_\ell$$

$$x = x'_k e'_k = x'_k a_{k\ell} e_\ell$$

$$\Rightarrow x_\ell = x'_k a_{k\ell}$$

$$\text{or} \quad x'_k = a^{-1}_{k\ell} x_\ell$$

map between vector spaces

A map between two vector spaces can be expressed as a matrix

$$A : V \to W \leftarrow f_k$$

e_k ↑ n dimensional

m dimensional

$$A e_k = \sum a_{k\ell} f_\ell$$

n x m matrix

Kernel of a map

$$\ker A = \{ x \in V \mid A(x) = 0 \}$$

$$\operatorname{Im} A = \{ y \in W \mid y = A(x),\ x \in V \}$$

$$\dim \ker A + \dim \operatorname{Im} A = \dim V$$

Lecture # 4

Last time:
- action principle
- Noether's theorem
- linear vector spaces

Today: Review of linear algebra

Dual space

V vector space V^* dual space

V^* is maps $f : V \to \mathbb{F}$

numbers

$$f(x) = f(x_\mu e_\mu) = x_\mu f(e_\mu) = x_\mu f_\mu$$

components of $f \in V^*$

e_μ basis of V

e^*_μ basis of V^* $e^{*\mu} e_\nu = \delta_{\mu\nu}$

So $f = \sum_\mu f_\mu e^{*\mu}$

inner product

$V \times V \longrightarrow F$

$$\langle x, y\rangle = \langle y, x\rangle^*$$

$$\langle x, \lambda y + \mu z\rangle = \lambda \langle x, y\rangle + \mu \langle x, z\rangle$$

$$\langle \lambda x + \mu y, z\rangle = \lambda^* \langle x, z\rangle + \mu^* \langle y, z\rangle$$

when F is $\mathbb{R}$ then $\langle x, y\rangle = \langle y, x\rangle$ and $\lambda^* = \lambda$

$\langle e_\mu, e_\nu \rangle = g_{\mu\nu}$

a basis is <u>orthonormal</u> if $g_{\mu\nu} = \delta_{\mu\nu}$

if $F = \mathbb{R}$ and $\tilde{f} \in V^*$

i.e. $\tilde{f}$ is a linear map $V \to \mathbb{R}$.

then $\exists\ f \in V$ such that

$\tilde{f}(x) = \langle f, x \rangle$

if $x = x_\mu e_\mu$

$$\text{then} \quad \tilde{f}(x_\mu e_\mu) = x_\mu \tilde{f}^\mu = x_\mu \langle \underbrace{f}_{e_\nu f_\nu}, e_\mu \rangle$$

$= x_\mu f_\nu g_{\nu\mu}$

$$\Rightarrow \tilde{f}_\mu = \underbrace{g_{\nu\mu}}_{\substack{\shortparallel \\ g_{\mu\nu}}} f_\nu$$

<u>Euclidean vectors</u> ; $\mathbb{R}^n$ $g_{\mu\nu} = e_\mu \cdot e_\nu$

$x = x^\mu e_\mu$ $\qquad x_\mu = (x \cdot e_\mu) = g_{\mu\nu} x^\nu$

lowering operator

Bra and Ket

$x \in V$ then $\langle x, \cdot \rangle \in V^*$

$$\langle x, y \rangle \to \mathbb{C}$$
$$y \in V$$

anti-linear

$$\langle (\lambda + \mu) x, \cdot \rangle = \lambda^* \langle x, \cdot \rangle + \mu^* \langle x, \cdot \rangle$$

$$|\psi\rangle \to |\psi\rangle^+ = \langle \psi|$$

↑ dual vector

$$(\lambda |\psi\rangle + \mu |\chi\rangle)^* = \lambda^* \langle \psi| + \mu^* \langle \chi|$$

<u>inner product</u>: $\langle \psi | \chi \rangle = (\langle \psi|, |\chi\rangle)$

<u>conjugate map</u>: $A : V \to W$

map induced by A $\quad A^* : W^* \to V^*$

$$x \in V \qquad f \in W^*$$

then

$$A x \in W$$
$$f(A(x)) \in \mathbb{C}$$
$$\llcorner \in V$$

$$\Rightarrow f(A(\cdot)) \in V^*$$

$$A^* : f \to f(A(\cdot))$$

linear map $f : V \to \mathbb{C}$

then $\exists$ $\underline{f} \in V$ such that $f(x) = (\underline{f}, x)$

$$f(x) = f(x^\mu e_\mu) = x^\mu f(e_\mu) = x^\mu f_\mu$$

$$(\underline{f}, x) = (f^\mu e_\mu, x^\nu e_\nu)$$

$$= f^{\mu *} x^\nu g_{\mu\nu}$$

true for all x^ν $\Rightarrow$ $f^{\mu *} g_{\mu\nu} = f_\nu$

$x \to (y, Ax)$ is a linear map

$\Rightarrow$ $\exists z \in V$ such that $(z, x) = (y, Ax)$

define the Hermitian adjoint

$$z = A^+ y$$

$$\Rightarrow \quad (A^+ y, x) = (y, Ax)$$

choose $x = e_\mu$, $y = e_\nu$

$$(y, Ax) = (e_\nu, A e_\mu) = (e_\nu, a_{\mu\rho} e_\rho)$$
$$= a_{\mu\rho} g_{\nu\rho}$$

$$(A^+ y, x) = (A^+ e_\nu, e_\mu)$$
$$= (\hat{a}_{\nu\rho} e_\rho, e_\mu) = \hat{a}^*_{\nu\rho} g_{\rho\mu}$$

orthonormal basis $g_{\nu\rho} = \delta_{\nu\rho}$

$$\Rightarrow \hat{a}^*_{\nu\mu} = a_{\mu\nu}$$

or $\hat{a}_{\mu\nu} = a^*_{\nu\mu}$

<u>Direct sum of vector spaces</u>

U, V vector spaces

$$U \oplus V = \{ \lambda (\underset{\substack{\cap \\ U}}{u_1}, \underset{\substack{\cap \\ V}}{v_1}) + \mu (\underset{\substack{\cap \\ U}}{u_2}, \underset{\substack{\cap \\ V}}{v_2})$$

Quotient spaces

$U \in W$ complementary space V
if $W = U \oplus V$

V is not unique.

quotient space is unique

W/U : $x = y \mod U$ if $x - y \in U$

this is a coset and the set of equivalent elements is called an equivalence classs

$A : U \to V$
co-kernel $V/\mathrm{Im}\, A$

orthogonal complement
$$U^\perp = \{x \in W \mid (x,y) = 0 \ \forall y \in U\}$$

$W = U \oplus U^\perp$

$\dim W/U = \dim U^\perp = \dim W - \dim U$

Projection operator

$P^2 = P \qquad P : V \to V$

if $x \in \operatorname{Im} P$ then $\exists\, y \mid x = Py$

the $Px = P^2 y = Py = x$

$Px = 0 \Rightarrow x = 0$

only $x = 0$ is in $\ker P$ and $\operatorname{Im} P$

So $V = \operatorname{Ker} P \oplus \operatorname{Im} P$

Linear equations

$Ay = b \qquad A \; m \times n$ matrix

$m < n$ solution is not unique

$m > n$ solution may not exist

rank of a square matrix

$\dim \operatorname{Ker} A = \dim \operatorname{Ker} A^+$

$\Rightarrow$ column rank of a square matrix is equal to the row rank

(21)

$x \in \ker A \Rightarrow (y, Ax) = 0 \quad \forall y$

$\qquad\qquad\qquad\quad \| $

$\qquad\qquad\quad (A^+ y, x)$

$\Rightarrow x \perp \operatorname{Im} A^+$

$x \perp \operatorname{Im} A^+$ ~~then~~ $(x, A^+ y) = 0 \quad \forall y \in V$

$\Rightarrow (Ax, y) = 0 \quad \forall y \in V$

$\ker A = (\operatorname{Im} A^+)^\perp$

$\ker A^+ = (\operatorname{Im} A)^\perp$ (just start with A)

$\dim \ker A + \dim \operatorname{Im} A = \dim V$

$\dim \ker A^+ + \dim \operatorname{Im} A^+ = \dim V$

$\dim \ker A = \dim (\operatorname{Im} A^+)^\perp$

$\qquad = \dim V - \dim \operatorname{Im} A^+$

$\qquad = \dim \ker A^+$

$\Rightarrow \dim \ker A = \dim \ker A^+$

Fredholm alternative

I. Either $Ax = b$ has a unique solution or $Ax = 0$ has a solution

$$\dim \ker A = \dim \mathrm{Im}\, A^{\perp}$$

II If $Ax = 0$ has n independent solutions, then also $A^{+}x = 0$ has

$$\dim \ker A = \dim \ker A^{+}$$

III. If II holds then $Ax = b$ has no solution unless

$b \perp$ all solutions of $A^{+}x = 0$

if $b \parallel$ solution of $A^{+}x = 0$

then $A^{+} b = 0$

$\Rightarrow b \in \ker A^{+} = \mathrm{Im}\, A^{\perp}$

$\Rightarrow b = 0$