Solutions of Homework set number 7

1. a) There is one eigenvector (0,0,0) which cannot be normalized So V^{-1} does not exist for diagonalization $A = V^{-1} \Lambda V$

b) The matrix is not diagonalizable

$a_0 \mathbb{1} + a_1 A + a_2 A^2 = 0$ only has the solution $a_0 = a_1 = a_2 = 0$. So, the minimum order polynomial equation is of 3rd order.

It is; $\mathbb{1} + -\frac{1}{4} A + \frac{1}{50} A^2 - \frac{1}{2000} A^3 = 0$

the roots of $1 - \frac{1}{4} x + \frac{1}{50} x^2 - \frac{1}{2000} x^3 = 0$

are given by $x = 10, 10, 20$,

They are repeated, so the matrix is not diagonalizable

c) The Jordan canonical form is

given by $A_J = \begin{pmatrix} 10 & 1 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 20 \end{pmatrix} = V^{-1} A V$

$\Rightarrow A V = V^{+1} A_J$ let $V = (v_1, v_2, v_3)$

$A(v_1, v_2, v_3) = (v_1, v_2, v_3) \begin{pmatrix} 10 & 1 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 20 \end{pmatrix} = (10 v_1, v_1 + 10 v_2, 20 v_3)$

$\Rightarrow A v_1 = 10 v_1$
$A v_2 = v_1 + 10 v_2$
$A v_3 = 20 v_3$

$(A - 10\mathbb{1}) v_2 = v_1$
$(A - 10\mathbb{1}) v_1 = 0$
$\Rightarrow v_2 \in \operatorname{Ker}(A - 10\mathbb{1})^2$

$\Rightarrow V = \begin{pmatrix} -1 & 1 & -3 \\ -3 & 0 & 1 \\ 2 & 0 & 6 \end{pmatrix}$ $(A - 10\mathbb{1})^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$

2. $f_n(x) = x + \frac{x^2}{n}$

a) Does not converge uniformely on $\mathbb{R}$

b) On $[-2,2]$ then $|f_n(x) - \cancel{f}x| = \frac{x^2}{n} < \varepsilon$

if $n > \frac{4}{\varepsilon}$.

c) $|f_n(x) - f_m(x)| = x^2 \left|\frac{1}{n} - \frac{1}{m}\right|$

not a Cauchy sequence on $\mathbb{R}$

but a Cauchy sequence on $[-2,2]$

then $n > \frac{4}{\varepsilon}$ and $m > \frac{4}{\varepsilon}$

$|f_n(x) - f_m(x)| < \varepsilon$

3. $f_n(x) \to f(x)$ on $[a,b]$ uniformely

$$\lim_{n\to\infty} \int_a^b f_n(x) \overset{?}{=} \int_a^b f(x)\,dx$$

We look at $\int_a^b |f_n(x) - f(x)|\,dx$

$\forall \varepsilon, \exists N$ such that $|f_n(x) - f(x)| < \varepsilon$ on $[a,b]$

$\Rightarrow \int_a^b |f_n(x) - f(x)| < (b-a)\varepsilon$ for $n > N$

and $\int_a^b (f_n(x) - f(x))\,dx < \int_a^b |f_n(x) - f(x)|\,dx$

$< \varepsilon(b-a)$ for $n > N$