Solution of HW6

2) $x^2 - 2y^2 + z^2 + 4xy - 6xz + 2yz$

$$= (x+2y-3z)^2 - (4y^2 + 9z^2 - 12yz) + z^2 - 2y^2 + 2yz$$

$$= (x+2y-3z)^2 \cancel{-2} - 6\left(y \cancel{+} - \tfrac{7}{6}z\right)^2 + \underbrace{\tfrac{49}{6}z^2 - 8z^2}_{\frac{1}{6}z^2}$$

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$$(T-\lambda)^p = 0 \quad \text{for } p = N$$

T $N \times N$ matrix

i) $T\phi = \lambda_0 \phi \qquad (T-\lambda)^N \phi = (\lambda_0 - \lambda)^N \phi = 0$

$$\Rightarrow \lambda_0 = \lambda$$

if T can be diagonalized, then $(T-\lambda)^p$ can be diagonalized but $(T-\lambda)^N = 0$ and $(T-\lambda)^p \neq 0$ for $p<N$

This is not possible. For a diagonal form

ii) $(T-\lambda)^p \neq 0,\ p < N \Rightarrow \exists e_1 \mid (T-p)^{N-1} e_1 \neq 0$

then $(T-p)^{N-2} e_1 \neq 0$ because if zero also

$$(T-p)\,(T-p)^{N-2} e_1 = 0, \text{ etc.}$$

$$(T-p)^N = 0 \Rightarrow (T-p)^N e_1 = 0$$

iii) $e_k = (T-\lambda)^{k-1} e_1 \quad a_3 \quad a_N (T-\lambda)^{N-1} e_1$

assume $a_2 (T-\lambda) e_1 + (T-\lambda)^2 e_k + \cdots$ act wit $T-\lambda$

$\Rightarrow a_1 e_1 = 0$, act wit

then $\overset{(T-\lambda)}{a_1 e_1} + a_2 (T-\lambda)^2 e_1 + \dots + a_{N-1}(T-\lambda)^{N-1} = 0$

etc

$\Rightarrow$ all e_k are linear dependent

$$\Rightarrow (T-\lambda) e_1 = \alpha e_1$$

$$\Rightarrow (T-\lambda)^N e_1 = \alpha^N e_1$$

which is incorrect $\Rightarrow$ qed.

iv) $e_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \dots \quad e_N = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$(T-\lambda) e_1 = e_2 \quad (T-\lambda) e_2 = e_3 \quad (T-\lambda) e_{N-1} = e_N$

$$T-\lambda = \begin{pmatrix} 0 & 1 & \dots & & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & & \\ 0 & & & 0\ 1 & 0 \\ 0 & \dots & & 0\ 0 & 1 \\ 0 & \dots & \dots & & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} \lambda & 1 & 0 & \dots & & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & 0 & \dots 0 \\ & & & & & \\ 0 & \dots & & & 0 & \lambda \end{pmatrix}$$

3. $\vec{a} \times \vec{b} = \varepsilon_{ijk} \, a_j b_k$

rotation $a_j' = \mathcal{O}_{jj'} a_{j'}$

$b_k' = \mathcal{O}_{kk'} b_{k'}$

$\Rightarrow \quad \vec{a}' \times \vec{b}' = \varepsilon_{ijk} \mathcal{O}_{jj'} \mathcal{O}_{kk'} a_{j'} b_{k'}$

$\Rightarrow \quad \mathcal{O}_{i'i} (\vec{a}' \times \vec{b}')_{i'} = \underbrace{\varepsilon_{i'jk} \mathcal{O}_{i'i} \mathcal{O}_{jj'} \mathcal{O}_{kk'}}_{\det \mathcal{O} = 1 \text{ for rotatation}} a_{j'} b_{k'}$

$\mathcal{O}^T = \mathcal{O}^{-1}$

$= \varepsilon_{ij'k'} a_{j'} b_{k'}$

$\Rightarrow \quad (\vec{a}' \times \vec{b}')_{i'} = \mathcal{O}_{i'i} \, \varepsilon_{ij'k'} a_{j'} b_{k'}$

$(\vec{a}' \times \vec{b}')_{i'} = \mathcal{O}_{i'i} (a \times b)_i$

$\Rightarrow$ length of $|a \times b|$ is invariant under rotations