1

1a) $\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \ker A = \left\{ \begin{pmatrix} -b \\ a \end{pmatrix} \right\}$

1b) $\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a x + b y \\ \lambda a x \; \lambda b y \end{pmatrix} \Rightarrow \mathrm{Im}\, A = \left\{ \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right\}$

1c) $(\ker A)^{\perp} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\}$

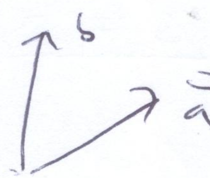
2) Ker A and Im A are independent

$\Rightarrow R_2 = \ker A + \mathrm{Im}\, A$

Also $\ker A \perp \ker A^{\perp} \Rightarrow R_2 = \ker A + \ker A^{\perp}$

by $\mathrm{Im}\, A \neq \ker A^{\perp}$

2b)



Make a rotation so that $\vec a$ and $\vec b$ are in the xy plane

$\vec a \parallel$ x axis

area spanned by $\vec a$ and $\vec b$ is $|\vec a||\vec b|\sin\phi$

outer product of $\vec a$ and $\vec b$

$$(|\vec a|, 0, 0) \times (|\vec b|\cos\phi, |\vec b|\sin\phi, 0)$$

$$= (0, 0, |\vec a||\vec b|\sin\phi)$$

3) $$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & \\ & d \end{pmatrix}\left(1 + \begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix}\right)$$

$$= \det a \det d \; \det\left(1 + \begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix}\right)$$

$$\parallel$$

$$e^{\mathrm{Tr}\log\left[\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix} + 1\right]}$$

$$\mathrm{Tr}\log\left[\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix} + 1\right] = \mathrm{Tr}\sum \frac{(-1)^{n+1}}{n}\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix}^n$$

use cyclicity of Trace $$= \mathrm{Tr}\sum \frac{(-1)^{2n+1}}{2n}\begin{pmatrix} a^{-1}b\,d^{-1}c & 0 \\ 0 & d^{-1}c\,a^{-1}b \end{pmatrix}^n$$

$$= +\frac{1}{2}\mathrm{Tr}\log\begin{pmatrix} 1 - a^{-1}b\,d^{-1}c & 0 \\ 0 & 1 - a^{-1}c\,d^{-1}b \end{pmatrix}$$

$$\Rightarrow e^{\mathrm{Tr}\log\left[\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix} + 1\right]}$$

$$= \det(1 - a^{-1}b\,d^{-1}c) \; \cancel{\det^{\frac{1}{2}}(1 - d^{-1}c\,a^{-1}b)}$$

$$\Rightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det d \det(a - b\,d^{-1}c)$$

3)

a) $H\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} cy \\ c^+x \end{pmatrix} = 0$

C is $m \times n$ matrix say $m \leq n$

$cy = 0$ m equations n unknowns

$c^+x = 0$ n equation m unknows

$m \leq n \Rightarrow x = 0$

$cy = 0$ has $n-m$ independen solutions

$\Rightarrow \ker H = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \middle| \ cy = 0 \right\}$

b) rank of H is # independent rows

for $m \leq n$ all rows of c are independent

but at most ~~n~~ rows of c^+ can be independent

$\Rightarrow$ rank ~~(~~ H) $= 2m$

We see that rank H + ker H = m+n