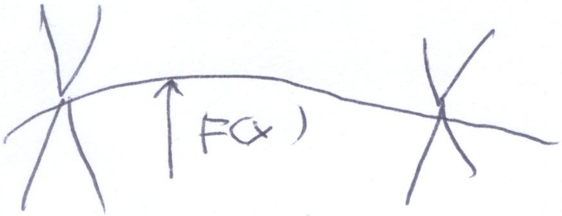
Solutions of HW 12

5.2



boundary value problem

$$\frac{d^4 y}{dx^4} = F(x) \qquad y(0) = y(1) = 0$$
$$y''(0) = y''(1) = 0$$

From 5.1 the boundary terms are

$$\frac{1}{2} K u^* v''' \Big|_0^L - \frac{1}{2} K u^{*\prime} v'' \Big|_0^L + \frac{1}{2} K u^{*\prime\prime} v' \Big|_0^L$$

$$- \frac{1}{2} K u^{*\prime\prime\prime} v \Big|_0^L \qquad L = 1$$

vanish by b.c of L

adjoint b.c $\quad u(L) = u(0) = 0$
$\qquad u''(L) = u''(0) = 0$

Same b.c. as for L $\Rightarrow$ L is self-adjoint

1) zero modes $y = a + bx + cx^2 + dx^3$

b.c. $\quad a = 0 \quad b + c + d = 0$

$y'' = 2c + \cancel{d} \, 6dx \quad \Rightarrow c = 0$
$\qquad 2c + 6dL = 0$
$\qquad \Rightarrow d = 0$

$\Rightarrow b = 0$

$\Rightarrow$ no zero modes

c) $\partial_x^4 G(x, \xi) = \delta(x - \xi)$

$\Rightarrow \partial_x^2 G(x,\xi)$, $\partial_x G(x,\xi)$

and $G(x,\xi)$ must be

continuous at $x = \xi$

$\partial_x^3 G(x,\xi)$ must jump by 1 at $x = \xi$

boundary condition

$$G(0,\xi) = G(L,\xi) = 0$$

$$\partial_x^2 G(x,\xi)\Big|_{x=0} = \partial_x^2 G(x,\xi)\Big|_{x=L} = 0$$

d) $x \neq \xi$ $\partial_x^4 G(x,\xi) = 0$

$\Rightarrow$ $x < \xi$ $G_< = a_< + b_< x + c_< x^2 + d_< x^3$

$x > \xi$ $G_> = a_> + b_> x + c_> x^2 + d_> x^3$

$\partial_x^3 G_< = 6 d_<$ $\Rightarrow d_> = d_< + \frac{1}{6}$

$\partial_x^3 G_> = 6 d_>$

$G(0,\xi) = 0 \Rightarrow$ $a_< = 0$ V

$G(L,\xi) = 0$ $a_> + b_> L + c_> L^2 + d_> L^3 = 0$

$G''(0,\xi) = 0$ $c_< = 0$ V

$G''(L,\xi) = 0$ $2 c_> + 6 d_> L = 0$

$\partial_x^2 G(x,\xi)$ $x < \xi$ $2 c_< + 6 d_< x$

$x > \xi$ $2 c_> + 6 d_> x$

5b

Should be continuous $c_< = 0$

at $x=\xi$ $\Rightarrow$ $6 d_< \xi = 2 c_> + \underset{\substack{\| \\ d_< + \frac{1}{6}}}{d_>} 6 \xi$

$\Rightarrow$ $c_> = -\frac{1}{2}\xi$ ✓

$\partial_x G(x,\xi)$ is continuous

$x<\xi$ $\quad b_< + 2 c_< x + 3 d_< x^2$

$x>\xi$ $\quad b_> + 2 c_> x + 3 d_> x^2$

$\Rightarrow$ $c_< = 0$

$b_< + 3 d_< \xi^2 = b_> + 2 c_> \xi + 3\xi^2 (d_< + \frac{1}{6})$

$\Rightarrow b_< = b_> + -\xi^2 + \frac{1}{2}\xi^2$

$\Rightarrow b_> = b_< + \quad + \frac{1}{2}\xi^2$

$G(x,\xi)$ is continuous

$\Rightarrow$ $b_< \xi + d_< \xi^3 = a_> + \underset{\substack{\| \\ b_< + \frac{1}{2}\xi^2}}{b_>} \xi + (-\frac{1}{2}\xi)\xi^2 + (d_< + \frac{1}{6})\xi^3$

$0 = a_> \quad -\frac{1}{2}\xi^3 + \frac{1}{2}\xi^3 + \frac{1}{6}\xi^3$

$\Rightarrow$ $a_> = \quad -\frac{1}{6}\xi^3$ ✓

$c_> = -\frac{1}{2}\xi \qquad d_> = -\frac{1}{3L} c_> = \frac{1}{6L}\xi$

we know $\quad a_< = 0$ ✓ $\quad c_< = 0$ ✓

$d_< = d_> - \frac{1}{6}$ $\quad c_> = -\frac{1}{2}\xi$ ✓ $\quad d_> = \frac{1}{6L}\xi$ ✓

$= \frac{1}{6L}\xi - \frac{1}{6}$ ✓ $\quad a_> = -\frac{1}{6}\xi^3$ ✓

(5.2)

$b_>$ follows from $\qquad L=1$

$$a_> + b_> L + c_> L^2 + d_> L^3 = 0$$

then $b_< = b_> \qquad + \frac{1}{2}\xi^2$

So all constants are determined uniquely

e) $x < \xi$

$$G(x,y) = \overset{0}{a_<} + b_< x + d_< x^2$$

$x > \xi$

$$= -\xi^2 + \frac{1}{2}\xi^3 + (b_< + \xi - \frac{1}{2}\xi^2) x - \frac{1}{2}\xi x^2 + (d_< + \frac{1}{6}) x^3$$

$x < \xi \qquad b_< x + d_< x^3$

$x > \xi \qquad b_< x + d_< x^3 - \xi^2 - \frac{1}{2}\xi^3 + \xi x - \frac{1}{2}\xi^2 x$

$$- \frac{1}{2}\xi x^2 + \frac{1}{6} x^3$$

(5.2)

$b_>$ follows from

$$a_> + b_> + c_> + d_> = 0$$

and $b_>$ From $b_< = b_> + \frac{1}{2}\xi^2$

This gives $$b_< = \frac{1}{6}\left(2\xi - 3\xi^2 + \xi^3\right)$$

$$b_> = \frac{1}{6}\left(2\xi + \xi^3\right)$$

The Green's function is given by

$x<\xi$ $$G_<(x,\xi) = \frac{1}{3}x\xi - \frac{1}{2}x^2\xi + \frac{1}{6}x^3\xi - \frac{\xi^3}{6} + \frac{x\xi^3}{6}$$

$x>\xi$ $$G_>(x,\xi) = \frac{1}{3}x\xi - \frac{x^3}{6} + \frac{x^3\xi}{6} - \frac{x\xi^2}{2} + \frac{x\xi^3}{6}$$

the $G_<(x,\xi) = G_>(\xi,x)$

f) Solution is

$$y = \int_0^x d\xi\, G_<(x,\xi)\, F(\xi)\, d\xi + \int_x^1 d\xi\, G_>(x,\xi)\, F(\xi)\, d\xi$$

trivial

5.6

$T_1 \quad y_0 = 0 \quad y_{N+1} = 0$

$T_2 \quad y_0 = 0$

in the last row
we should have $\quad -y_{N-1} + 2y_N - y_{N+1}$
but we have $\quad -y_{N-1} + y_N$

$$\Rightarrow \quad y_N - y_{N+1} = 0 \quad \Rightarrow \quad y'(1) = 0$$

b)
$$\min(i,j) = \begin{pmatrix} 1 & 1 & 1 & \cdots & & 1 \\ 1 & 2 & 2 & \cdots & & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 \\ 1 & 2 & 3 & 4 & 4 \cdots & 4 \\ & & & & & K-1 \; K-1 \\ 1 & 2 & 3 & 4 & 5 \cdots & K-1 \; K \end{pmatrix}$$

$$T_1 \min(i,j) = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \cdots & 0 \\ & & & & & K-1 \; K+1 \\ 1 & 2 & 3 & 4 & \cdots & \end{pmatrix}$$

For $T_2 \min(i,j)$ the last row
is $(0, 0, 0, \cdots 0, 1)$ and all
other rows are the same

$$\Rightarrow (T_2)^{-1} = \min(i,j)$$

$(T_1)_{mi}\, ij =$

$= T_{1\,m\,m-1}\, m-1\, j + T_{1\,m\,m}\, mj + T_{1\,m\,m+1}\, m+1\, j$

$= -j(m-1) + 2jm - j(m+1)$

$= 0$

when $m \neq 1$ or $m \neq K$

$m = 1$

$T_{1i}\, ij = 2j - j2 = 0$

$m = K$ $\quad T_{Ki}\, ij = -(K-1)j + 2Kj = Kj + j$

<u>last row</u>

$T_{Ki}\, \frac{ij}{K+1} = 1, 2, 3, \ldots K$

and $K+1-K = 1$

$\Rightarrow (T^{-1})_{ij} = \min(i,j) - \frac{ij}{K+1}$

c) continuum operator is $-\partial_x^2$

$y(0) = y(1) = 0$

$-\partial_x^2 (a + bx) = 0$

$G(x, \xi) = bx \quad x < \xi \Rightarrow G(x,\xi) = \alpha_1 x(1-\xi)$

$G(x, \xi) = c(1-x) \quad x > \xi \Rightarrow G(x,\xi) = \alpha_2 \xi(1-x)$

continuity $\quad b\xi = c(1-\xi)$

$\Rightarrow \alpha_1 = \alpha_2$

$x<\xi \quad \partial_x G(x,\xi) = \alpha_1(1-\xi)$

$x>\xi \quad \partial_x G(x,\xi) = -\alpha_2\xi = -\alpha_1\xi$

jump should be $-1 \Rightarrow -\alpha_1\xi - \alpha_1(1-\xi) = -1$

$\alpha_1 = +1$

5.6

$\Rightarrow G(x,\xi) = +x(1-\xi) \quad x<\xi$

$G(x,\xi) = +\xi(1-x) \quad x>\xi$

$$\begin{cases} = +x - x\xi & x<\xi \\ = +\xi - x\xi & x>\xi \end{cases}$$

min ij

$\frac{ij}{n+1}$

Tc: $y(0) = 0 \qquad y = x$

$y'(1) = 0 \qquad y = a$

$\Rightarrow G(x,\xi) = x \quad , x<\xi$

$a \quad , x>\xi$

$a = \xi$ by continuity.

$\partial\, \partial_x G(x,\xi)\Big|_{x=\xi} = -1$ ok.

$\Rightarrow G(x,\xi) \longrightarrow \min(i,j)$

$$R_{\omega^2} = (-\partial_x^2 - \omega^2)^{-1}$$

$$(-\partial_x^2 - \omega^2)\left(\frac{1}{\omega \sin \omega L} \sin \omega x \sin \omega(L-\xi)\right)$$

$$= 0 \qquad x < \xi$$

$$(-\partial_x^2-\omega^2)\frac{1}{\omega\sin\omega L}\sin\omega(L-x)\sin\omega\xi \Rightarrow$$

$$x > \xi$$

$$\partial_x R_{\omega^2}\Big|_{x=\xi+0} - \partial_x R_{\omega^2}\Big|_{x=\xi-0}$$

$$= \frac{-1}{\omega\sin\omega L}\,\omega \cos(L-\xi)\omega \sin\omega\xi - \frac{\omega \cos\omega\xi \sin\omega(L-\xi)}{\omega\sin\omega L}$$

$$= \frac{1}{\sin\omega L}\left(-\sin(\omega\xi + (L-\xi)\omega)\right) = -\frac{\sin\omega L}{\sin\omega L} = -1$$

$\Rightarrow R_{\omega^2}(x,\xi)$ is resolvent

$-\partial_x^2$ has eigenfunction $\sin\frac{\pi n}{L}x$

eigenvalue $+\frac{\pi^2 n^2}{L^2} = \omega_n^2$

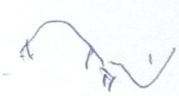
$\sin \omega_n L = 0$ at theses values.

$\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi n x}{L}$ $\qquad \int_0^L \sin^2\frac{\pi n}{L} = \frac{L}{2}$

c) $\sin\omega L = \sin\omega_n L + (\omega-\omega_n)L\cos\omega_n L$

$$\frac{1}{\omega_n(\omega-\omega_n)L\cos\omega_n L}\sin\omega_n x \sin\omega_n(L-\xi) - \frac{\sin\omega_n x \sin\omega_n(L-\xi)}{\omega_n(\omega+\omega_n)L\cos\omega_n L}$$

$$\frac{2\cancel{\omega_n}}{\cancel{\omega_n}(\omega^2-\omega_n^2)L\cos\underbrace{\omega_n L}_{\pi n}} \sin\omega_n x \sin\omega_n(L-\xi)$$



$$\sin(\pi n - \omega_n\xi) = (-1)^{n+1}\sin\omega_n\xi$$

$$\cos \pi n = (-1)^n$$

$$= +\frac{2}{L}\frac{\sin\omega_n x \sin\omega_n L}{\omega_n^2-\omega^2}$$