3.2 $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = \dfrac{e^x}{1+x^2}$

Solution of homogeneous eq. $y = e^{\alpha x}$

$$\alpha^2 e^{\alpha x} - 2\alpha e^{\alpha x} + e^{\alpha x} = 0$$

$$\Rightarrow \alpha^2 - 2\alpha + 1 = 0 \Rightarrow \alpha = 1$$

$y_1 = e^x$ is a solution

$$\frac{dW}{dx} = 2W \Rightarrow W = e^{2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix}$$

$$= e^x y_2' - e^x y_2 = e^{2x}$$

$$\Rightarrow y_2 = x e^x$$

By substitution you see that

$$-\tfrac{1}{2} e^x \log(1+x^2) + x e^x \arctan x$$

is special solution

$\Rightarrow$ general solution is as given in the problem

Homework set # 8

3.5) $\left(-\frac{1}{2}\frac{d^2}{dx^2}+(V-E)\right)\psi=0$

$x = x(t\,z)$ $\qquad \hat{\psi}(z)=\psi(t(z))$

$\Rightarrow \cancel{\frac{d\psi}{dx}} \qquad \frac{d}{dz}\hat{\psi}(z)$

$= \psi' \frac{dx}{dz}$

$$\frac{d^2}{dz^2}\hat{\psi}(z) = \underbrace{\psi''}_{2(V-E)\psi(x(z))}\left(\frac{dx}{dz}\right)^2 + \underbrace{\psi'}_{\psi'(x(z))}\frac{d^2x}{dz^2}$$

use eqs (3.41) and (3.42)

$$\Rightarrow -\frac{1}{2}\frac{d^2}{dz^2}\hat{\psi} + (V-E)\hat{\psi} + \frac{1}{2}\underbrace{\psi'}_{\hat{\psi}'(z)\left(\frac{dx}{dz}\right)^{-1}}\frac{d^2x}{dz^2} = 0$$

$$\Rightarrow \partial_{z^2}\hat{\psi} + (-2)(V-E)\hat{\psi} - \hat{\psi}'(z)\left(\frac{dx}{dz}\right)^{-1}\frac{d^2x}{dz^2} =$$

$$\Rightarrow \Omega = -2(V-E) + \frac{1}{2}\frac{d}{dz}\left(\left(\frac{dx}{dz}\right)^{-1}\frac{d^2x}{dz^2}\right) - \frac{1}{4}\,\left(\frac{dx}{dz}\right)^{-2}\left(\frac{d^2x}{dz^2}\right)^2$$

$$\Rightarrow \Omega = -2(V-E) + \frac{1}{2}\left(\frac{dx}{dz}\right)^{-1}\frac{d^3x}{dz^3} - \frac{1}{2}\frac{x''^2}{x'^2}$$

$$-\frac{1}{4}\frac{1}{x'^2}x''^2$$

$$= -2(V+-E) + \frac{1}{2}\frac{x'''}{x'} - \frac{3}{4}\frac{x''^2}{x'^2}$$

$\Rightarrow$ qed.

3.5b) $x \to z \to w$

after the $x \to z$ transformation the $-\frac{1}{4}\{x, z\}$ term becomes part of the potential and for $z \to w$ it transforms as $\left(\frac{dz}{dw}\right)^2\left(-\frac{1}{4}\{x, z\}\right)$

so the additional terms we are getting

$$\left(\frac{dz}{dw}\right)^2\left(-\frac{1}{4}\{x, z\}\right) - \frac{1}{4}\{z, w\}$$

we can also make the transformation in one step and get $-\frac{1}{4}\{x, w\}$

$$\Rightarrow \left(\frac{dz}{dw}\right)^2 \{x, z\} + \{z, w\} = \{x, w\}$$

for the $V - E$ term we use

$$\frac{dx}{dz}\frac{dz}{dw} = \frac{dx}{dw}$$

3) $-\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$

$y = \hat{y}\,w \qquad \frac{dy}{dx} = \frac{d\hat{y}}{dx}w + \hat{y}\frac{dw}{dx}$

$\frac{d^2y}{dx^2} = \frac{d^2\hat{y}}{dx^2}w + 2\frac{d\hat{y}}{dx}\frac{dw}{dx} + \hat{y}\frac{d^2w}{dx^2}$

$\Rightarrow -\frac{d^2\hat{y}}{dx^2}w - 2\frac{d\hat{y}}{dx}\frac{dw}{dx} - \hat{y}\frac{d^2w}{dx^2} + 2xw\frac{d\hat{y}}{dx} + 2x\hat{y}\frac{dw}{dx} = 0$

$\Rightarrow -2\frac{dw}{dx} + 2xW = 0$

$\Rightarrow \frac{dw}{dx} = x\,w \quad \Rightarrow w = c\,e^{x^2/2}$

choose $c = 1$

$\frac{d^2w}{dx^2} = w + x\frac{dw}{dx} \qquad e^{x^2}$

$\Rightarrow \cancel{-\frac{d^2\hat{y}}{dx^2}e^{x^2} + 2xe^{x^2}\frac{d\hat{y}}{dx} + 2x\hat{y}\,2xw''} = 0$

$\Rightarrow \cancel{-\frac{d^2y}{dx^2} + 2x\frac{d\hat{y}}{dx}}$

$-\frac{d^2\hat{y}}{dx^2}e^{x^2/2} - \hat{y}\left(w + x\frac{dw}{dx} - 2x\frac{dw}{dx}\right) = 0$

$\Rightarrow -\frac{d^2\hat{y}}{dx^2}e^{x^2/2} - e^{\frac{1}{2}x^2}\hat{y} + \hat{y}x^2e^{x^2/2} = 0$

$\Rightarrow \frac{d^2\hat{y}}{dx^2} = \hat{y}(x^2-1)$

$\Rightarrow \hat{y}_1 = e^{-\frac{1}{2}x^2}$

$\hat{y}_2 = e^{-\frac{1}{2}x^2}\int_{x_0}^{x} e^{y^2}\,dy$ see hw 10