Solutions of homework set # 10

1.

$$\lim_{\varepsilon\to 0}\int \frac{1}{x+i\varepsilon}\,\varphi(x)\,dx = \lim_{\varepsilon\to 0}\int \frac{x-i\varepsilon}{x^2+\varepsilon^2}\,\varphi(x)\,dx$$

$$= \lim_{\varepsilon\to 0}\int \frac{x}{x^2+\varepsilon^2}\,\varphi(x) - \lim_{\varepsilon\to 0}\, \underbrace{i\varepsilon\,\frac{\left(\varphi(0)+x\,\varphi'(0)\right)+\dots}{x^2+\varepsilon^2}}_{\text{all higher order terms vanish for } \varepsilon\to 0}$$

$$= \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\left[\left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right)\frac{x}{x^2+\varepsilon^2}\,\varphi(x)\,dx + \underbrace{\int_{-\delta}^{\delta}\frac{x}{x^2+\varepsilon^2}\,\varphi(x)\,dx}_{\mathcal{O}\left(\frac{\delta^3}{\varepsilon^2}\right)\to 0}\right] - i\int\frac{\varepsilon\,dx}{x^2+\varepsilon^2}\,\varphi(0)$$

Taylor $\quad = -\pi i\,\varphi(0)$

(take $\delta\to 0$ first

can interchange $\lim_{\delta\to 0}$ and $\lim_{\varepsilon\to 0}$

$$\Rightarrow \quad P\left(\frac{1}{x}\right) - \pi i\,\delta(x)$$

Solutions of Homework set #7

3.1 a) $y_1 = u(x)$ is solution of $y'' + V(x)\,y = 0$

Trial solution $y = u v$

$$y' = u'v + uv'$$
$$y'' = u''v + 2u'v' + uv''$$
$$= -Vuv + 2u'v' + uv''$$

diff. eq.: $\cancel{-Vuv} + 2u'v' + uv'' + \cancel{Vuv} = 0$

$$\Rightarrow \quad 2u'v' + uv'' = 0$$

$$\Rightarrow \quad 2\frac{u'}{u} = -\frac{v''}{v'}$$

$$\Rightarrow \quad 2\frac{d\log u}{dx} = -\frac{d\log v'}{dx}$$

$$\Rightarrow \quad \log u^2 = \log (v')^{-1} + c'$$

or $u^2 = \dfrac{c}{v'} \Rightarrow$

$$\frac{dv}{dx} = \frac{c}{u^2} \Rightarrow v = \int^x \frac{c}{u^2}\,dx'$$

$\Rightarrow$ second solution $y_2 = u(x)\displaystyle\int^x \frac{c}{u^2(x')}\,dx'$

Wronskian
$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} u & u\int^x \frac{c\,dx}{u^2} \\ u' & u'\int^x \frac{c\,dx}{u^2} + \frac{c}{u} \end{vmatrix}$$

$$= \cancel{uu'\int^x \frac{c\,dx}{u^2}} + c - \cancel{uu'\int^x \frac{c\,dx}{u^2}}$$

only vanishes if $c = 0$, but then $y_2 = 0$ which is the trivial solution

3.13 $\quad y_1 y_2 = c$

$\Rightarrow \quad y_1'' + 2y_1'y_2' + y_2'' = 0 \quad , \quad y_1'y_2 + y_1y_2' = 0$

$-p_1 y_1' - p_2 y_1 + 2y_1'y_2' - p_1 y_2' - p_2 y_2 = 0$

Wronskian $\quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

$= y_1 y_2' + y_1 y_2'$

$= 2 y_1 y_2'$

$\frac{dW}{dx} = -\frac{p_1}{p_0} W = -p_1 W$

$\Rightarrow \quad 2y_1'y_2' + 2y_1 y_2'' = -p_1 2 y_1 y_2'$

$\Rightarrow \quad y_1'y_2' + y_1(\cancel{-p_1 y_2'} - p_2 y_2) = \cancel{-p_1 y_1 y_2'}$

$\Rightarrow \quad y_1'y_2' - p_2 y_1 y_2 = 0$

$\Rightarrow \quad y_1'y_2' = p_2 c$

$\Rightarrow \quad y_1''y_2' + y_1'y_2'' = p_2' c$

$\Rightarrow \quad (-p_1 y_1' - p_2 y_1) y_2' + y_1'(-p_1 y_2' - p_2 y_2) = p_2' c$

$\Rightarrow \quad -p_1 (y_1'y_2') 2 - p_2 (y_1 y_2' + y_1' y_2) = p_2' c$

$\Rightarrow \quad -p_1 p_2 2c - p_2 0 = p_2' c$

$\Rightarrow \quad 2p_1 p_2 = p_2'$

3.1c

$$y'' + \frac{x}{(x+1)x^2} y' - \frac{(x+1)^2}{x^2} y = 0$$

$$\Rightarrow P_1 = \frac{1}{x(x+1)} \qquad P_2 = \frac{(x+1)^2}{x^2}$$

$$P_2' = \frac{2(x+1)}{x^2} + \frac{2(x+1)^2}{x^3} = \frac{2(x+1)}{x^2}\left(1 - \frac{(1+x)}{x}\right)$$

$$= 2\frac{(x+1)}{x^3} = -2 P_1 P_2$$

$$\frac{dW}{dx} = -\frac{P_1}{P_0} W = -\frac{W}{x(x+1)}$$

$$\Rightarrow \frac{dW}{W} = -\frac{dx}{x(x+1)} = dx\left(-\frac{1}{x} + \frac{1}{x+1}\right)$$

$$= d(\log(x+1) - \log x)$$

$$\Rightarrow W = c\frac{x+1}{x}$$

$$\Rightarrow y_1 y_2' - y_2 y_1' = c\frac{x+1}{x}$$

Since the condition of D is satisfied, let us look for solutions with $y_1 y_2 = c_2$

$$\Rightarrow y_2 = \frac{c_2}{y_1} \qquad y_2' = -\frac{c_2}{y_1^2} y_1'$$

$$\Rightarrow y_1\left(-\frac{c_2}{y_1^2} y_1'\right) - \frac{c_2}{y_1} y_1' = c\frac{x+1}{x}$$

$$-c_2 \frac{y_1'}{y_1} 2 = c\frac{(x+1)}{x}$$

$$-2c_2 \frac{d}{dx}\log y_1 = c\left(\frac{x+1}{x}\right)$$

3

$$\left(-\frac{d^2}{dx^2} + 4x^2\right) y =$$

$$y = e^{-x^2} \longrightarrow \frac{dy}{dx} = -2x e^{-x^2}$$

$$\frac{d^2y}{dx^2} = -2e^{-x^2} + 4x^2 e^{-x^2}$$

$$\Rightarrow \left(-\frac{d^2}{dx^2} + 4x^2\right) y = 2e^{-x^2} = 2y$$

$$\Rightarrow \left(-\frac{d^2 \cancel{y}}{dx^2} + (4x^2 - 2)\right) y = 0$$

$$\Rightarrow P_0 = 1 \quad P_1 = 0, \quad P_2 = 4x^2 - 1$$

$\Rightarrow \frac{dW}{dx} = 0 \Rightarrow$ Wronskian does not depend on x

solutions $y_1 = e^{-x^2}$ an y_2

$$W = \begin{vmatrix} e^{-x^2} & y_2 \\ -2x e^{-x^2} & y_2' \end{vmatrix} = +2x y_2 e^{-x^2} + y_2' e^{-x^2}$$

W = constant w

$$\Rightarrow +W = e^{-x^2}(2x y_2 + y_2')$$

$$= e^{-x^2} e^{-x^2} \frac{d}{dx} e^{x^2} y_2$$

C

$$\Rightarrow \frac{d}{dx} e^{x^2} y_2 = W e^{2x^2}$$

$$\Rightarrow y_2 = \left(c + \int^x e^{2y^2} dy\right) e^{-x^2}$$

$$= \left(\int_{x_0}^x e^{2y^2} dy\right) e^{-x^2}$$

$$\frac{dy_2}{dx} = e^{x^2} + \left(\int_{x_0}^{x} e^{2y^2}\,dy\right)(-2x)e^{-x^2}$$

$$\frac{d^2y_2}{dx^2} = e^{2x x^2} - 2x e^{+x^2} + \int_{x_0}^{x} e^{2y^2}\,dy\,(-2)\,e^{-x^2}$$

$$+ \int_{x_0}^{x} e^{2y^2}\,dy\,(+4x^2)\,e^{-x^2}$$

$$= \cancel{-2xe^{x^2}} - 2y_2 + 4x^2 y_2$$

$$+ \cancel{e^{x^2}}$$

$$= (4x^2 - 2)\,y_2$$