Homework #1, Solutions

1.3 a) $L(y) = \frac{1}{y}\sqrt{1+y'^2}$ does not depend on x.

So we can use the First integral

$$L(y,y') - y'\frac{\partial L}{\partial y'} = c$$

$$\Rightarrow \frac{1}{y}\sqrt{1+y'^2} - \frac{y'}{y}\frac{y'}{\sqrt{1+y'^2}} = c$$

$$\Rightarrow 1+\cancel{y'^2} - \cancel{y'^2} = cy\sqrt{1+y'^2}$$

$$\Rightarrow \frac{1}{c^2y^2} = 1+y'^2 \Rightarrow y'^2 = \frac{1}{c^2y^2}-1$$

$$y' = \sqrt{\frac{1}{c^2y^2}-1} \Rightarrow \frac{dy}{\sqrt{\frac{1}{c^2y^2}-1}} = dx$$

$$\Rightarrow \frac{cy\,dy}{\sqrt{1-c^2y^2}} = c^{-1}d\left(-\sqrt{1-c^2y^2}\right) = dx$$

$$\Rightarrow \sqrt{1-c^2y^2} = cx+d$$

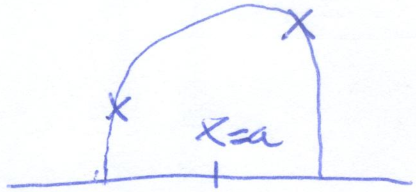
$$\Rightarrow 1-c^2y^2 = (cx+d)^2$$

$$y^2 = \frac{1}{c^2}\left(1-(cx+d)^2\right)$$

this is the equation of a circle.

$y>0$, so it is a semicircle

b)



$$y = \sqrt{R^2 - (x-a)^2}$$

the two parameters are determined by the two points

$$\left.\begin{array}{l} y_1^2 + (x_1-a)^2 = R^2 \\ y_2^2 + (x_2-a)^2 = R^2 \end{array}\right\} \Rightarrow y_1^2 - y_2^2 + (x_1-a)^2 - (x_2-a)^2 = 0$$

$$\Rightarrow y_1^2 - y_2^2 + x_1^2 - x_2^2 - 2a(x_1 - x_2) = 0$$

$$a = \frac{y_1^2 - y_2^2 + x_1^2 - x_2^2}{2(x_1 - x_2)}$$

R follows from the equation

c) $F_3(y) \to \infty$ for $y \to 0$

by continuity, these lines never meet q.

1.7) Poincaré disk

$$ds^2 = \frac{dx^2 + dy^2}{(1-x^2-y^2)^2}$$

$$S_{\text{ar}} = \int \frac{1}{1-x^2-y^2}\sqrt{\dot x^2+\dot y^2}\,dt$$

first integral $L - \sum_i y_i' \frac{\partial L}{\partial y_i'}$

$$\frac{\sqrt{\dot x^2+\dot y^2}}{1-x^2-y^2} - \frac{\dot x}{1-x^2-y^2}\frac{\dot x}{\sqrt{\dot x^2+\dot y^2}} - \frac{\dot y^2}{(1-x^2-y^2)\sqrt{\dot x^2+\dot y^2}} = \text{con}$$

$= 0$ So we cannot use it

$$\delta\sqrt{\dot x^2+\dot y^2} = \sqrt{\dot x^2+2\dot x\,\delta\dot x+\dot y^2+2\dot y\,\delta\dot y} - \sqrt{\dot x^2+\dot y^2}$$

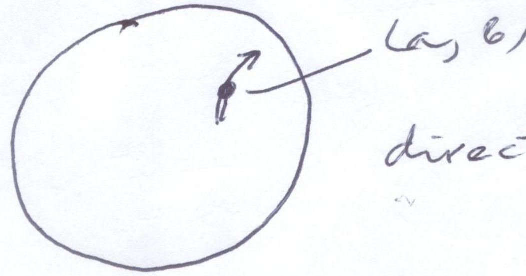
now use the hint

$$\delta x = \varepsilon y \qquad \delta y = -\varepsilon x$$

make ε time dependent

$$\delta S = \frac{2x\varepsilon y - 2y\varepsilon x}{(1-x^2-y^2)^2}\sqrt{\dot x^2+\dot y^2} + \frac{(\dot x\dot\varepsilon y + \cancel{\dot x\varepsilon\dot y} - \dot y\dot\varepsilon x - \cancel{\dot y\varepsilon\dot x})}{(1-x^2-y^2)\sqrt{\dot x^2+\dot y^2}}$$

partial integrate the $\dot\varepsilon$ term gives the desired result b)



direction $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

$$\frac{1}{1-x^2-y^2}\,\frac{x\dot{y}-y\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = \text{constant}$$

$$\Rightarrow \frac{1}{1-x^2-y^2}\,\frac{x\frac{\dot{y}}{\dot{x}}-y}{\sqrt{1+\frac{\dot{y}^2}{\dot{x}^2}}} = C$$

constant is determined by $x = a$, $y = b$, $\frac{\dot{y}}{\dot{x}} = \alpha$

we need two equation to determine $x(t)$ and $y(t)$

c) substitute

$$\frac{1}{1-R^2-2Rx_0\cos t-x_0^2}\,\frac{(R^2\cos^2 t+x_0R\cos t + R^2\sin^2 t)}{R} = c$$

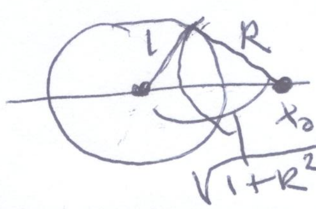
$$R + x_0\cos t = c\,(1-R^2-x_0^2-2Rx_0\cos t)$$

$$R = c\,(1-R^2-x_0^2)$$

$$\cancel{x_0} = -2Rc\cancel{x_0} \Rightarrow c = -\frac{1}{2R}$$

$$x_0^2 = 1-R^2-\frac{R}{c} = 1-R^2+2R^2 = 1+R^2$$

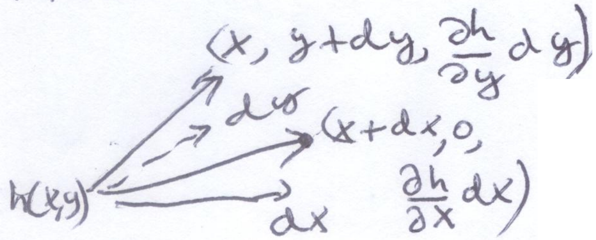
$$\Rightarrow x_0 = \sqrt{1+R^2}$$



Pythagoras valid $\Rightarrow$ qed

1.13)

Membrane



area is the outer product

$$\vec{A} = \left(0, dy, \frac{\partial h}{\partial y} dy\right) \times \left(dx, 0, \frac{\partial h}{\partial x} dx\right)$$

$$\vec{A} = \left(\frac{\partial h}{\partial x} dx\,dy, -\frac{\partial h}{\partial y} dx\,dy, dx\,dy \right)$$

$$|\vec{A}| = \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}\; dx\,dy$$

$$\Rightarrow A = \int dx\,dy \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}$$

b) For small distortions we can Taylor expand

$$A = \text{const} + \int dx\,dy \left((\partial_x h)^2 + (\partial_y h)^2 \right)$$

c) $\delta A \doteq 0 \Rightarrow \partial_x^2 h + \partial_y^2 h = 0$

d) $L = \int \left(\frac{1}{2} \rho_0 (\partial_t h)^2 - \frac{1}{2} T \left((\partial_x h)^2 + (\partial_y h)^2 \right) \right) dx\,dy$

eqs. of motion $\partial_t^2 h - \partial_x^2 h - \partial_y^2 h = 0$