1. Functionals

$$\mathcal{F}(f, f') \to \mathbb{R}$$

$f + \delta f$ gives $\frac{\delta \mathcal{F}}{\delta f}$

Euler Lagrange equation

$$S = \int L(f, f')\,dx$$

$$\delta S = 0 \Rightarrow \quad \frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} = 0.$$

First integral $\quad \frac{d}{dt}\left(L - f'\frac{\partial L}{\partial f'}\right) = 0$

Lagrange multiplier

constraint $\quad g(x, y) = 0, \quad G(f, f') = 0$

Action with constanx

$$S = \int \left(L(f, f') + \lambda G(f, f')\right) dx$$

In classical mechanics

$$\frac{d}{dL}\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.$$

Noether's theorem

Global symmetry $\theta \to \theta + \varepsilon \alpha$ then make α t dependent at extremum. This variation gives a conservation law

2. Vector spaces and linear algebra

- axioms
- basis e_k
- map between vector spaces $V \to W$

$$A e_k = A_{ke} e_e$$

Kernel of a map $\ker A$
Image of a map $\operatorname{Im} A$

$$\dim \ker A + \dim \operatorname{Im} A = \dim V$$

dual space $V^* = \{ f \mid f : V \to \mathbb{F} \}$

dual basis $e^{*\mu} e_\nu = \delta^\mu_\nu$

inner product $\langle x, y \rangle$

$\hat{f} \in V^*$ then $\exists\, f \in V \mid \hat{f}(x) = \langle f, x \rangle$

conjugate map:

$$A : V \to W$$
$$A^* : W^* \to V^*$$

$$A^* : \quad f \to f(A(\cdot))$$

Hermition adjoint:

$$\langle A^+ y, x \rangle = \langle y, A x \rangle$$

get matrix if x and y are basis elem.

Direct sum

Quotient space W/U

$x \in W/U \qquad y \in W/U$

$x \sim y$ is $x - y \in U$

Cokernel $\quad V/\operatorname{Im} A$

Orthogonal complement:

$$U^\perp = \{ x \in W \mid (x, y) = 0 \ \forall y \in U \}$$

Projection operator: $\quad P^2 = P$

$$\ker P \cap \operatorname{Im} P = 0$$

<u>Linear equations</u> $\quad Ax = b$

Fredholm alternative

rank of a matrix

<u>Determinants</u>

$$\det A = \sum \varepsilon_{i_1 \dots i_n} a_{1 i_1} \cdots a_{n i_n}$$

- $\det A$ is linear in each column and each row
- changes sign under the interchange of two rows or two column
- $\det A^T = \det A$

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

$\uparrow$ determinant of matrix with row i and column j deleted.

$$\sum_j a_{ij} C_{ji'} = \delta_{ii'} \det A$$

Adjugate matrix $(\text{Adj } A)_{ij} = C_{ji}$

$$A^{-1} = \frac{1}{\det A} \text{Adj}$$

<u>characteristic equation</u> $\det(A - \lambda 1) = 0$

Cayley's theorem.

$A = \{a_{ij}\}$

$$\frac{d}{da_{ij}} \log \det A = A^{-1}_{ji}$$

$A x^k = \lambda_k x^k$

$A^+ = A \Rightarrow \lambda_k \in \mathbb{R}$

$\left.\begin{array}{l} \lambda_k \neq \lambda_e \\ A = A^+ \end{array}\right\} \Rightarrow (x_k, x_e) = 0$

A is diagonalizable if there is a similarity transformation such that

$$V A V^{-1} = \begin{pmatrix} \lambda_1 \lambda_2 & 0 \\ 0 & \ddots \lambda_n \end{pmatrix}$$

A matrix A is not diagonalizable if the lowest order polynomial equation that A satisfies has multiple roots.

A matrix can always be transformed to a Jordan canonical form by a similarity transformation, eg

$$A = \begin{pmatrix} \lambda_1 & 1 & & & & \\ & \lambda_2 & 1 & & \Theta & \\ & & \lambda_3 & 1 & & \\ & & & \lambda_4 & & \\ & \Theta & & & \lambda_5 & \\ & & & & & \lambda_6 \end{pmatrix}$$

<u>quadratic forms</u> $\quad Q(x) = x^T A x$

Lagrange method

symplectic form $\quad \omega(x, y) = -\omega(y, x)$

bilinear

3. <u>Function spaces</u>

- Norm
- convergence
  - pointwise
  - uniform
  - in the mean

Banach space, Hilbert space

Inner product $\quad L^2$ norm

Cauchy-Swartz $\quad |(f, g)| \le \|f\| \, \|g\|$

orthonormal function sets $(u_n, u_m) = \delta_{nm}$

Orthogonal polynomials

$$(P_n, P_m) \sim \delta_{nm}$$

$$P_n = \sum_{k=0}^{n} a_k x^k$$

approximation: $\quad f \overset{n}{=} \sum_{k=1}^{N} a_k u_k$

Parcival $\quad \|f\|^2 = \sum |a_k|^2$

Three step recursion

Legendre, Hermite, Tsebychev polynomials

Distribution and test functions

$$f(x) = \int \delta(x-y)\, f(y)$$

δ' $\qquad \delta(a + xb) = \frac{1}{|b|}\, \delta(x + \frac{a}{b})$

weak derivative $\quad \int v(x)\, \varphi(x) = -\int u(x)\, \varphi'(x)$

v is the weak derivative of u

$$\frac{d}{dx} \log |x| = P\left(\frac{1}{x}\right)$$

principle value integral