

Linear differential equations

Homogeneous differential equation

$$Ly = 0$$

$$Ly = P_1(x)y^{(1)} + \dots + P_n(x)y$$

nth order diff. eq.

In homogeneous differential eqs:

nth order diff. eq. is equivalent to a set of n first order equations

$$y_1 = y$$

$$y_2 = y'$$

$$y_3 = y''$$

$$\vdots$$
$$y_n = y^{(n)}$$

then $y_2 = y_1'$

$$y_3 = y_2'$$

\vdots

$$y_n = -\frac{1}{P_n} (P_1 y_{n-1} + \dots + P_n y)$$

Flow equations

$$X^i(x^1, x^2, \dots, x^n, t)$$

vector field.

$$\frac{dx^i}{dt} = X^i(x_1, \dots, x_n, t)$$

initial condition $X^i(t=0) = X_0^i$

It is clear that there is a unique solution

(12)

The first order flow equations can also be combined in a single n 'th order equation in the t derivatives

The solutions of the n 'th order equation are a vector space

y_1, y_2 solutions $\Rightarrow ay_1 + by_2$ is a solution

There are n independent solutions

consider solutions with the initial conditions

$$y_1(0) = 1 \quad y^{(k)}(0) = 0 \quad k > 1$$

$$y^{(2)}(0) = 1 \quad y^{(k+2)}(0) = 0$$

$$y^{(n)}(0) = 1 \quad y^{(k \neq n)}(0) = 0$$

now if they are not linear independent then \exists

$$\lambda_k : \lambda_1 y_1 + \dots + \lambda_n y_n = 0$$

$$\Rightarrow \lambda_1 = 0$$

also any derivative $\lambda_1 y_1^{(p)} + \dots + \lambda_n y_n^{(p)} = 0$

$$\Rightarrow \lambda_p = 0$$

\Rightarrow all $\lambda_k = 0 \Rightarrow y_1, \dots, y_n$ are linearly independent

Wronskian

To find out if solutions are linearly independent, we calculate the Wronskian

$$W(y_1, \dots, y_n; x) = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ y_1^{(2)} & \dots & y_n^{(2)} \\ \vdots & \dots & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Differentiating a determinant is differentiating each row or column.

$$\Rightarrow \frac{dW}{dx} = \begin{vmatrix} y_1 & \dots & y_n \\ y_1' & \dots & y_n' \\ \vdots & \dots & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

all other terms vanish.

$$\text{and } y_k^{(n)} = -\frac{p_1}{p_0} y_k^{(n-1)} - \dots - \frac{p_n}{p_0} y_k$$

only this term $-\frac{p_1}{p_0} y_k^{(n-1)}$ gives a nonzero contribution

$$\Rightarrow \frac{dW}{dx} = -\frac{p_1}{p_0} W - \int_{x_0}^x \frac{p_1(x')}{p_0(x')} dx'$$

$$\Rightarrow W(x) = W(x_0) e^{-\int_{x_0}^x \frac{p_1(x')}{p_0(x')} dx'}$$

Liouville's theorem

w vanishes at all x or never because an exponential function is always nonzero.

Theorem: The Wronskian vanishes for linearly dependent functions

$$\lambda_1 y_1(x) + \dots + \lambda_n y_n(x) = 0$$

$$\Rightarrow \lambda_1 y_1^{(k)}(x) + \dots + \lambda_n y_n^{(k)}(x) = 0$$

This has a nontrivial solution

$$\Rightarrow W = 0$$

converse y_1, \dots, y_n solutions of an n th order ODE with $W(y_k, x_0) = 0$

Then $\exists \lambda_k$ not all zero such

$$\text{that } \lambda_1 y_1(x) + \dots + \lambda_n y_n(x) \equiv \psi(x)$$

$$\text{and } \psi(x_0) = \psi'(x_0) = \dots = \psi^{(n-1)}(x_0) = 0$$

These are linear eqs. For the λ_k and have a nontrivial solution if $W(y_k, x_0) = 0$

Normal Form

(5)

For polynomials $a_0 x^n + a_1 x^{n-1} + \dots + a_n$

it is in normal form if $a_1 = 0$

This is achieved by $x \rightarrow x - \left(\frac{a_1}{a_0}\right) \frac{1}{n}$

For differential equations

$$P_0 y^{(n)} + P_1 y^{(n-1)} + \dots + P_n y = 0$$

the normal form is when $P_1 = 0$

To find it we substitute $y = w \tilde{y}$

then
$$y^{(n)} = w \tilde{y}^{(n)} + n w' \tilde{y}^{(n-1)} + \binom{n}{2} w'' \tilde{y}^{(n-2)}$$

So we require $P_0 n w' + P_1 = 0$

$$\Rightarrow w' = -\frac{P_1}{n P_0}$$

$$\Rightarrow w(x) = w_0 e^{-\frac{1}{n} \int_0^x \frac{P_1(x')}{P_0(x')} dx'}$$

Normal form of second order equation

$$P_0 y'' + P_2 = 0 \Rightarrow y'' + \frac{P_2}{P_0} y = 0$$

This is like a Schrödinger equation

In homogeneous equations

(16)

$$P_0(x) y^{(n)} + P_1(x) y^{(n-1)} + \dots + P_n(x) = f(x)$$

Guess a solution y_G

Then general solution is $y_G + y_H$

with y_H is solution of the Homogeneous equation (also called complementary function)

$$P_0(x) y_H^{(n)} + \dots + P_n(x) = 0.$$

Example



$$R \frac{dQ}{dt} + \frac{Q}{C} = V$$

Homogeneous equation $R \frac{dQ}{dt} + \frac{Q}{C} = 0$

$$\Rightarrow Q = c_1 e^{-\frac{t}{RC}}$$

$Q = CV$ is a solution of the inhomogeneous equation

\Rightarrow general solution is $Q(t) = CV + c_1 e^{-\frac{t}{RC}}$

initial condition $Q(0) = 0 \Rightarrow c_1 = -CV$

$$\Rightarrow Q(t) = CV (1 - e^{-\frac{t}{RC}})$$

Variation of parameters

This is a trick to solve the inhomogeneous equation invented by Lagrange

We start from a solution of the homogeneous equation

$$y = \sigma_1 y_1 + \dots + \sigma_n y_n$$

the σ_k are functions of x and the y_k are independent solutions.

$$y' = \sigma_1 y_1' + \dots + \sigma_n y_n' + \underbrace{(\sigma_1' y_1 + \sigma_2' y_2 + \dots + \sigma_n' y_n)}_{\text{choose } \sigma_k' \text{ such that this vanishes}}$$

then $y'' = \sigma_1 y_1'' + \dots + \sigma_n y_n'' + \underbrace{(\sigma_1' y_1' + \dots + \sigma_n' y_n')}_{\text{choose } \sigma_k' \text{ such that this vanishes}}$

continues $y^{(n)} = \sigma_1 y_1^{(n)} + \dots + \sigma_n y_n^{(n)} +$

$$+ (\sigma_1' y_1^{(n-1)} + \sigma_2' y_2^{(n-1)} + \dots + \sigma_n' y_n^{(n-1)}) = \frac{F(x)}{P_0(x)}$$

$$\Rightarrow P_0 y^{(n)} + P_1 y^{(n-1)} + \dots + y = P_0 (\sigma_1 y_1^{(n)} + \sigma_2 y_2^{(n)} + \dots) + F(x) + P_1 (\sigma_1 y_1^{(n-1)} + \sigma_2 y_2^{(n-1)} + \dots)$$

$$\leftarrow P_n (\sigma_1 y_1 + \sigma_2 y_2 + \dots)$$

\Rightarrow this is a solution in homogeneous of the diff. eq

we still have to solve for the σ_k

(68)

$$\sigma_1' y_1 + \dots + \sigma_n' y_n = 0$$

$$\sigma_1' y_1^{(n-1)} + \dots + \sigma_n' y_n^{(n-1)} = \frac{f(x)}{p_0}$$

linear equations
they have a solution if $\begin{vmatrix} y_1 & \dots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0$

This is the case because we assumed that the solutions are independent.

Example

$$\frac{dy}{dx} + p(x)y = f(x)$$

homogeneous equation $\frac{dy}{dx} + p(x)y = 0$

$$\text{solution } y_1 = e^{-\int_a^x p(s) ds}$$

substitute $y_1 v_1$

$$y_1' v_1 + p(x) y_1 v_1 + y_1 v_1' = f(x)$$

require that $y_1 v_1' = f(x)$

$$\Rightarrow v_1' = \frac{f(x)}{y_1} \Rightarrow v_1 = e^{-\int \frac{f(x)}{y_1} dx}$$

then $y_1 v_1$ is solution of inhomogeneous equation.

Singular point:

$$P_0(x) y^{(n)} + P_1(x) y^{(n-1)} + \dots$$

singular point if $P_0(a) = 0$

This completely change the solutions
It often happens that P_0 vanishes at the end of an interval.

Generally this is a complicated subject and we will only look at a second order example:

$$P_0 y'' + P_1 y' + P_2 = 0$$

$$P_0(x) = (x-a)^2 P(x)$$

$$P_1(x) = (x-a) Q(x)$$

$$P_2(x) = R(x)$$

P, Q, R smooth and nonzero P, Q

$x=a$ is a regular singular point

close to $x=a$ the equation can be approximated by

$$P(a)(x-a)^2 y'' + Q(a)(x-a) y' + R(a) y$$

solution $y = (x-a)^\lambda$

$$P(a) \lambda(\lambda-1) + Q(a) \lambda + R(a) = 0$$

(70)

Two solutions $y_1 = (x-a)^{x_1} f_1(x)$
 $y_2 = (x-a)^{x_2} f_2(x)$

By substituting this back into the differential equation we get equations for f_1 and f_2