Function spaces

$C^n[a,b]$ n fold differentiable functions on $[a,b]$

$C^\infty[a,b]$ Analytic function on $[a,b]$

A function space is a linear vector space

$\lambda f(x) + \mu g(x)$ is also a function

To find the distance between two functions we need a norm. There are many different norms, eg
$\int_a^b |f(x)|\, dx$

A norm should satisfy:

i) positivity $\|f\| \geq 0$, $\|f\| = 0$ implies $f = 0$

ii) triangle inequality $\|f+g\| \leq \|f\| + \|g\|$

iii) linear homogeneity $\|\lambda f\| = |\lambda|\,\|f\|$

i) $\int_a^b |f(x)|\,dx$ satisfies i)

ii) $\int_a^b |f(x)+g(x)|\,dx \leq \int_a^b (|f(x)| + |g(x)|)\,dx = \|f\| + \|g\|$

iii) is obvious

Convergence

We often have to approximate a function $f(x)$ by a sequence $f_n(x)$ on $\mathcal{D}$

There are different ways of converging.

i) <u>Pointwise converge</u>: $\forall x \in \mathcal{D}$, the sequence $f_n(x)$ converges to $f(x)$

ii) <u>Uniform convergence</u>

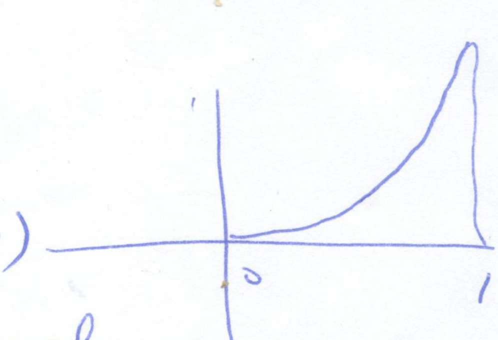
$$\sup_{x \in \mathcal{D}} |f_n(x) - f(x)| \to 0 \quad \text{for } n \to \infty$$

iii) <u>Convergence in the mean</u>

$$\int_{\mathcal{D}} |f_n(x) - f(x)|\, dx \to 0 \quad \text{for } n \to \infty$$

Uniform convergence is important because it is a condition for interchanging integrals and limits.

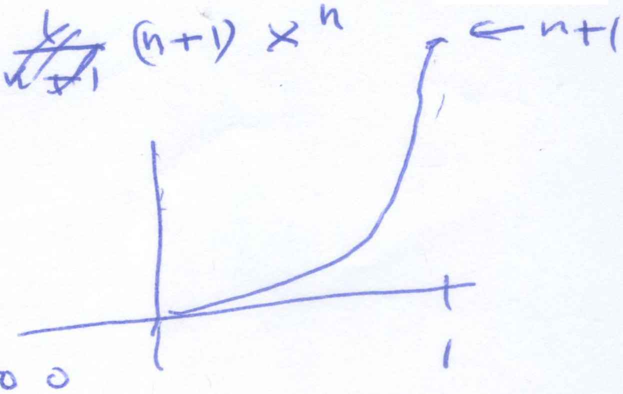
<u>Example 1)</u> $f_n = x^n$



- $f_n \to 0$ pointwise on $[0, 1)$
- does not converge uniformely

one $[0, 1]$ is does also not converge pointwis but converges in the mean

Example 2 $\quad f_n(x) = \cancel{\frac{1}{n+1}}(n+1)x^n$



$$\int_0^1 f_n(x) = 1$$

- converges pointwise to 0 on $[0,1)$
- does not converges uniformely on $[0,1)$

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$$

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0$$

Almost all notion

With convergene in the Lebesgue sense an function converges to a function in almost all pints, i.e. $\forall x \in D$ with the exception of a set of zero measure.

norm for Lebesgue Space

L^p norm

$$\|f\|_p = \left(\int |f(x)|^p dx\right)^{1/p}$$

Most common norm $\quad p = 2$

Complete space

Cauchy sequence

$\forall \varepsilon > 0 \quad \exists N$ such that $\forall n, m > N \quad \|f_n - f_m\| < \varepsilon$

a normed vector space is complete if each Cauchy sequence converges to some element of the space.

Banach space a normed complete vector space

$L^p[a,b]$ space are complete when the norm is interpreted as Lebesgue integral

Hilbert space Banach space $L^2[a,b]$

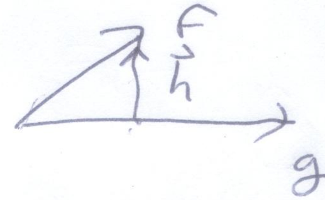
in general, a Hilbert space is a Banach space where the norm is derived from the inner product.

For $L^2[a,b]$
$$\|f\| = \left(\int_a^b f\,f(x)\,dx\right)^{1/2}$$
$$= (f,f)^{1/2}$$

proof that $\|f\|_2$ is a norm

Cauchy - Schwartz inequality

$|(f,g)| \leq \|f\| \, \|g\|$



define $h = f - \frac{(f \cdot g)\, g}{(g \cdot g)}$

th $(h \cdot g) = 0 \Rightarrow h \perp g$.

apply Pythagoras to h and $\frac{(f,g)}{(g,g)} g$

$$\|f\|^2 = \|h\|^2 + \frac{|(f,g)|^2 \, \|g\|^2}{\|g\|^4}$$

$$\Rightarrow |(f,g)|^2 \leq \|f\|^2 \|g\|^2$$

Triangle inequality

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + 2(f,g)$$
$$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\,\|g\|$$
$$= (\|f\| + \|g\|)^2$$

of course $\|f\| \geq 0$ and $\|\lambda f\| = |\lambda| \, \|f\|$

and if $\|f\| = 0$ then $f = 0$

Consequence of Cauchy-Schwartz inequality

$$\|f_n - f\| \to 0$$

then $|(f_n, g) - (f, g)|$

$$= |((f_n - f), g)| \le \|f_n - f\| \, \|g\|$$

$$\Rightarrow (f_n, g) \to (f, g)$$

<u>Orthonormal function sets</u>

$$(u_n, u_m) = \delta_{nm}$$

<u>Example</u> a) $u_n = e^{2\pi i n x}$

then $\int_0^1 dx \, (u_n, u_m) = \delta_{nm}$

complex inner product

b) $u_m = \sqrt{2} \sin m\pi x$

if u_n is a complete set on (a, b) then any function f in the Hilbert space can be expanded as

$$f = \sum_{n=0}^{\infty} a_n u_n(x)$$

$$(u_n, f) = \sum_{n=0}^{\infty} a_m (u_n, u_m) = a_m$$

Orthogonal polynomials

inner product $(u,v) = \int_a^b w(x)\, u(x)\, v(x)\, dx$

polynomials depend on w

$$P_0(x) = \frac{1}{\|1\|_w}$$

$$P_n(x) = \sum_{k=0}^{n} a_k x^k$$

orthonormal polynomials $(P_n, P_m) = \delta_{nm}$

Construction

$$P_{n+1} = \alpha x P_n + \sum_{k=0}^{n} a_k P_k$$

$$(P_{n+1}, P_k) = 0 \quad \forall k \le n$$

$k \le n \qquad \alpha(P_k, xP_n) + a_k = 0$

$$\Rightarrow \quad P_{n+1} = \alpha x P_n + \sum_{k=0}^{n} (-\alpha)(P_k, xP_n) P_k$$

α follows from normalization of P_{n+1}

$$(P_{n+1}, P_{n+1}) = 1 = \alpha^2 (xP_n, xP_n) - 2\alpha^2 \sum_{k=0}^{n} (P_k, xP_n)^2 + \sum_{k=0}^{n} \alpha^2 (P_k, xP_n)^2$$

Best Approximation

$$\Delta = \|f - \sum_{n=1}^{N} a_n u_n\|^2$$

← orthonormal basis

we minimize Δ

$$\Delta = \|f\|^2 - \sum_n a_n (f, u_n) - \sum a_n^* (u_n, f)$$

$$+ \sum a_m^* a_n \underbrace{(u_m, u_n)}_{\delta_{mn}}$$

$$\uparrow$$

$$\sum a_n^2$$

$$\Rightarrow \Delta = \|f\|^2 + \sum_{n=1}^{n} |a_n - (u_n, f)|^2 - \sum_{n=1}^{N} |(f, u_n)|^2$$

this is minimized if $a_n = (u_n, f)$

Parcival's theorom

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

$$\|f\|^2 = \int f^*(x) f(x)\, dx$$

$$= \sum_{nm} \int a_n^* a_m u_n^* u_m \, dx$$

$$= \sum |a_n|^2$$

<u>Example</u> $u_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ on $[-\pi, \pi]$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{i y x}$$

$$\|f\| = \int_{-\pi}^{\pi} \frac{1}{2\pi}\, dx = 1$$

$$\frac{1}{\sqrt{2\pi}} e^{i y x} = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i y x - inx} = \frac{\sin \pi (y-n)}{\pi (y-n)}$$

<u>Parceval</u> ; $\sum_n \frac{\sin^2 \pi (y-n)}{(\pi(y-n))^2} = 1$

$$\Rightarrow \sum_n \frac{\sin^2 \pi y}{(\pi(y-n))^2} = 1$$

$$\Rightarrow \sum_n \frac{1}{\pi^2 (y-n)^2} = \frac{1}{\sin^2 \pi y}$$

Three step recursion relation

Orthogonal polynomials satisfy a 3 step recursion relation

$$x P_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x)$$

Proof $(x P_n, P_k) = (P_n, x P_k) = 0$ for $k \leq n-2$

$\Rightarrow$ qed

Weierstrass approximation theorem

$f(x)$ on $[a, b]$ continuous then $\exists$ polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon \quad \forall x \in [a, b]$

$\Rightarrow$ Orthogonal polynomials are complete

So any function can be expressed in terms of an (infinite) sum of orthogonal polynomials.

Example of orthogonal polynomials

Legendre polynomials

$[-1, 1] \qquad w(x) = 1$

The Legendre polynomial satisfy the Rodriguez formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Proof i) P_n is a polynomial of order n

ii) orthogonality $\quad k < n$

$$\int_{-1}^{1} \frac{1}{2^n n!} \frac{1}{2^k k!} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^k}{dx^k}(x^2-1)^k$$

- partial integrate $k+1$
- boundary terms vanish because of (x^2-1)

$= 0$ for $n > k$

for $n = k$ we get

$$\frac{(2n)!}{2^{2n} n! \, n!} \int (x^2-1)^n (-1)^n dx = \frac{2}{2n+1}$$

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Recursion

$$(2n+1)\, x P_n = (n+1) P_{n+1} + n P_{n-1}$$

example $P_0 = 1$ $\qquad P_1 = ax + b$

$$\int_{-1}^{1} (ax+b) P_0 = 0 \qquad b + a\tfrac{1}{2} 0 = 0$$

$$\Rightarrow b = 0$$

$$\int_{-1}^{1} (ax)^2 dx = \frac{2}{3} = a^2 \frac{1}{3} 2 \Rightarrow \cancel{a = \sqrt{\tfrac{3}{2}}}$$

$$a = 1$$

$$P_2 = ax^2 + bx + c$$

can do the same but simpler to use Rodriguez formula

$$P_2(x) = \frac{1}{4.2} \frac{d^2}{dx^2} \underbrace{(x^2-1)^2}_{x^4 - 2x^2 + 1}$$

$$= \frac{1}{8}(12x^2 - 4) = \frac{3}{2}x^2 - \frac{1}{2}$$

Let us see if recursion work

$n = 1$

$$3 x P_1 = 2 P_2 + P_0$$

$$3 x x = (6x^2 - 1) + 1$$

Hermite polynomials

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \sim \delta_{nm}$$

we will see that this follows from:

Generating Function $\displaystyle\sum_{n=0}^{\infty} H_n \frac{t^n}{n!} = e^{2tx - t^2}$

From generating function

$$H_n(x) = \frac{d^n}{dt^n} e^{2tx - t^2} \Big|_{t=0}$$

$$= \frac{d^n}{dt^n} e^{-(t-x)^2 + x^2} \Big|_{t=0}$$

$$= (-1)^n \left[\frac{d^n}{dx^n} e^{-(t-x)^2} \right] e^{x^2} \Big|_{t=0}$$

$$= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Orthogonality relation

$$\int H_n H_m e^{-x^2} dx = 2^n n! \sqrt{\pi} \delta_{nm}$$

proof choose $n > m$

$$\int (-1)^{n+m} e^{-x^2} e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \underbrace{e^{x^2} \frac{d^m}{dx^m} e^{-x^2}}_{m\text{'th order polynomial}}$$

So the integral vanishes by partial integration.

for $n = m$: $$\int dx \left(\frac{d^n}{dx^n} e^{-x^2}\right) \underbrace{e^{x^2} \frac{d^n}{dx^n} e^{-x^2}}_{n\text{'th order polynomial}}$$

highest power is obtained by differentiating the exponent n times

$$= \int dx \left(\frac{d^n}{dx^n} e^{-x^2}\right)\left((-2x)^n + \text{lower order}\right)$$

partially integrate n time $= \int dx\, e^{-x^2} 2^n n!$

$$= 2^n n! \sqrt{\pi}$$

Tchebychef polynomials

First kind $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} U_n(x) U_m(x) \sim \delta_{nm}$

Second kind $\int_{-1}^{1} dx \sqrt{1-x^2}\, T_n(x) T_m(x) \sim \delta_{nm}$

They follow from the orthogonality of $\cos m\theta$ and $\sin n\theta$

$$\int_0^{\pi} \cos n\theta \cos m\theta \, d\theta \sim \delta_{nn}$$

$x = \cos\theta \qquad dx = -\sin\theta \, d\theta$

$\qquad\qquad\qquad\; = -\sqrt{1-x^2}\, d\theta$

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \underbrace{\cos(n \cos^{-1} x)}_{T_n(x) \text{ is polynomial because } \cos nx = \sum_{k=0}^{n} a_k \cos^k x} \cos m(\cos^{-1} x) \sim \delta_{nn}$$

$$\int_0^{\pi} d\theta \sin n\theta \sin m\theta \sim \delta_{nm}$$

$$= \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \underbrace{\sin(n\cos^{-1} x)}_{\text{not a polynomial.}} \sin(m\cos^{-1} x)$$

$$\sin \cos^{-1} x = \sqrt{1-\cos^2 \cos^{-1} x} = \sqrt{1-x^2}$$

$$= \int_{-1}^{1} dx \sqrt{1-x^2}\, \underbrace{\frac{\sin(n\cos^{-1} x)}{\sqrt{1-x^2}}}_{U_n(x) \text{ is polynomial}} \frac{\sin(m\cos^{-1}x)}{\sqrt{1-x^2}}$$

$\sin nx = \sum_{k=0}^{n} b_k \sin^k x \Longrightarrow$

Distribution and test functions

Linear operator in functionspace

$$f(x) = \int_a^b A(x,y)\, g(y)\, dy$$

δ - function

$$F(x) = \int \delta(x-y)\, f(y)\, dy$$

like identity operator

$\delta(x-y)$ is called a distribution and f is a test function which is a collection of smooth but otherwise undetermined functions

Derivative of δ function, $\delta'(x-y)$ which is also a linear operator in function space

$$\int \delta'(x-y)\, f(y)\, dy$$

$$\frac{d}{dx} = -\frac{d}{dy}$$

$$= (-)\int(-)\delta(x-y)\, f'(y) = f'(x)$$

part-int.

$$\Rightarrow \delta'(x-y) = \delta(x-y)\frac{d}{dy}$$

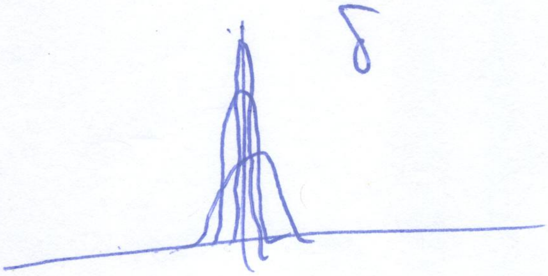
in general $\delta^{(n)}(x-y) = (-1)^n \delta(x-y) \frac{d}{dy}$

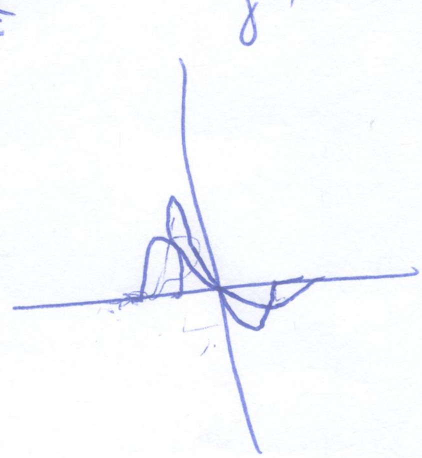
$$\delta'(-y) = -\delta'(y)$$

$$\delta^{(n)}(-y) = (-1)^n \delta(y)$$

$\delta(y)$ is even $\Rightarrow$ $\delta'(y)$ is odd.

<u>δ function as limit</u>





Distributions are functions with discontinuities and derivatives thereoff. They are defined by acting with them on a set of nice functions, called testfunction e.g infinitely differentiable functions that vanish sufficiently fast at ∞.

Distributions are in the dual space to the linear space of test functions

$(\delta, \varphi) = \varphi(0)$

$(\delta', \varphi) = -\varphi(0)$

$$\delta(ax-b) = \frac{1}{|a|}\delta\left(x-\frac{b}{a}\right)$$

proof

$$\int \delta(\underbrace{ax-b}_{y}) f(x)\,dx \qquad dy = |a|\,dx$$

$$= \int \frac{dy}{|a|}\delta(y)\, f\left(\frac{y+b}{a}\right) = \frac{1}{|a|} f\left(\frac{b}{a}\right)$$

weak derivative

$$\int v(x)\,\varphi(x)\,dx = -\int u(x)\,\varphi'(x)\,dx$$

then $v(x)$ is the weak derivative of $u(x)$

eg $\int \left[\frac{d}{dx}\operatorname{sign} x\right]\varphi(x) = -\int \operatorname{sign} x\ \varphi'(x)$

$$= +\int_{-\infty}^{-\varepsilon}\varphi'(x)\,dx - \int_{\varepsilon}^{\infty}\varphi'(x)\,dx$$

$$= \varphi(-\varepsilon) + \varphi(\varepsilon) \to 2\varphi(0)$$

$$= \frac{d}{dx}\operatorname{sign} x = 2\delta(x)$$

Example 2 weak derivative of $\log|x|$

$$I = -\int_{-\infty}^{\infty} \varphi'(x) \log|x| = \int_{-\infty}^{\infty} \varphi(x) \frac{d}{dx} \log|x|)$$

$$= -\int_{\varepsilon}^{\infty} \varphi'(x) \log x - \int_{-\infty}^{-\varepsilon} \varphi'(x) \log(-x)$$

$$= + \int_{\varepsilon}^{\infty} \varphi(x) \frac{1}{x} dx + \int_{-\infty}^{-\varepsilon} \varphi(x) \frac{1}{x} dx$$
$$+ \varphi(\varepsilon) \log \varepsilon - \varphi(\varepsilon) \log \varepsilon$$

$$= \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x} dx + \underbrace{\log \varepsilon \, (\varphi(\varepsilon) - \varphi(\varepsilon))}_{\to 0}$$

$$\equiv P\left(\frac{1}{x}\right) \varphi$$

principal value integral

only works if we choose the intervals symmetrically.