Vector space

$x, y \in V$ then $x + y \in V$

$\lambda x \in V$

τ scalar $\mathbb{R}, \mathbb{C}$

axioms

$$x + y = y + x$$
$$0 + x = x$$
$$\lambda(x + y) = \lambda x + \lambda y$$
$$(\mu + \lambda) x = \mu x + \lambda x$$
$$(x + y) + z = x + (y + z)$$
$$\lambda(\mu x) = (\lambda \mu) x$$
$$I x = x$$

finite dimensional vector space

n-dimensional vector space

$\exists n$ basis elements with $\lambda_1 e_1 + \cdots + \lambda_n e_n = 0$

implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

any element of V can be expressed in this basis

$$x = x_1 e_1 + \cdots + x_n e_n$$

x_k are unique

If e_k and e_k' are a basis, then

$$e_k' = a_{ke} e_e$$

$$x = \Sigma x'_k e'_k = \Sigma x_k e_k$$
$$= \Sigma x'_k a_{kl} l_e = \Sigma x_k e_k$$
$$= \Sigma x'_m a_{mk} e_k$$
$$\Rightarrow \quad x_k = x'_m a_{mk}$$

<u>map between vector spaces</u>

$$A : V \longrightarrow W$$

n dimensional e_k m dimensional f_k

$$A e_k = \sum_{l=1}^{m} a_{kl} f_l$$

a_{kl}: $n \times m$ matrix

If A is a linear map this is sufficient to calculate Ax for all x.

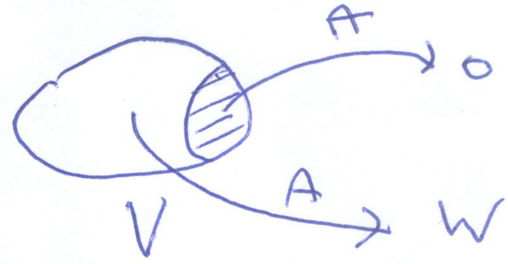
<u>Kernel of a map</u>

$$\text{Ker } A = \{ x \in V \mid A(x) = 0 \}$$

image: $\text{Im } A = \{ y \in W \mid y = A(x), x \in V \}$

$$\dim \text{Ker } A + \dim \text{Im } A = \dim V$$

__proof:__



Let us look at $x \in V$ that can not be written as $Ax \notin W/\{0\}$

$\Rightarrow \quad Ax = 0$ __qed__

Dual space:

V vector space

V^* dual vector space.

$V^* = \{ f \mid f : V \to \mathbb{F} \}$

f: linear. $\mathbb{F}$: numbers.

$f(x) = f(x_\mu e_\mu) = x_\mu f(e_\mu)$

$\Rightarrow f(e_\mu) = f_\mu \qquad \equiv x_\mu f_\mu$

f_μ: components of $f \in V^*$

e_μ basis of V

e_μ^* basis of V^* $\qquad e^{*\mu} e_\nu = \delta^\mu_\nu$

$f = \sum a_\mu e^{*\mu} \Rightarrow f(e_\nu) = \sum a_\mu e^{*\mu}(e_\nu)$
$= a_\nu$

$\Rightarrow a_\mu = f_\mu$

Inner product

$$V \times V \to \mathbb{F}$$
numbers

$$\langle x, y \rangle$$

$$\langle x, y \rangle = \langle y, x \rangle^*$$

$$\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$$

$$\langle \lambda x + \mu y, z \rangle = \lambda^* \langle x, z \rangle + \mu^* \langle y, z \rangle$$

if $\mathbb{F} = \mathbb{R}$ then $\langle x, y \rangle = \langle y, x \rangle$ and $\lambda^* = \lambda$.

$$\langle e_\mu, e_\nu \rangle = g_{\mu\nu}$$

if $g_{\mu\nu} = \delta_{\mu\nu}$ then the basis is orthonormal.

If $F = \mathbb{R}$ $\hat{f} \in V^*$

$\hat{f}$ is a linear map $V \to \mathbb{R}$

then $\exists$ $f \in V$ such that

$$\hat{f}(x) = \langle f, x \rangle.$$

<u>proof</u> Let $x = x_\mu e_\mu$

then $\breve{f}(x_\mu e_\mu) = x_\mu \breve{f}(e_\mu)$

$$= x_\mu \breve{f}^\mu$$

we want to write this as $\langle f, x \rangle$

$$\langle f, x \rangle = x_\mu \underbrace{\langle f, e_\mu \rangle}_{= e^\nu f_\nu}$$

$$= x_\mu f_\nu g^\nu{}_\mu$$

$$\Rightarrow \breve{f}^\mu = f_\nu g^\nu{}_\mu$$

$$\Rightarrow f = g^{-1} \breve{f}$$

Euclidean vectors: $\mathbb{R}^n$

$x = x^\mu e_\mu$ $\qquad g_{\mu\nu} = e_\mu e_\nu$

$$x_\mu = x \cdot e_\mu = g_{\mu\nu} e^\nu$$

lowering operator

Bra and Ket

$x \in V$ then $\langle x, \cdot \rangle \in V^*$

$\langle x, y \in V \rangle \longrightarrow \mathbb{C}$

anti linear $\langle (\lambda + \mu) x, \cdot \rangle = \lambda^* \langle x, \cdot \rangle + \mu^* \langle x, \cdot \rangle$

$|\psi\rangle \to |\psi\rangle^* = \langle \psi|$

dual vector

$$(\lambda |\psi\rangle + \mu |\chi\rangle)^\dagger = \lambda^* \langle \psi| + \mu^* \langle \chi|$$

inner product $\langle \psi | \chi \rangle = (\langle \psi|, |\chi\rangle)$

Conjugate map

$A : V \to W$

conjugate map $A^* : W^* \to V^*$

$x \in V$ $\qquad f \in W^*$

$Ax \in W \Rightarrow f(A(x)) \in \mathbb{C}$

$$\Updownarrow$$

~~A A~~ $f(A(\cdot)) : V \to \mathbb{C}$

$$\Uparrow \; V^*$$

$$A^* : f \to f(A(\cdot))$$

We use this concept to define the adjoint operator.

Adjoint operator:

linear map $f : V \to K$
then $\exists \check{f} \in V$ such that $f(x) = (\check{f}, x)$

$f(x) = f(x^\mu e_\mu) = x^\mu f(e_\mu) = x^\mu f_\mu$

$(\check{f}, x) = (\check{f}^\mu e_\mu, x^\nu e_\nu) = \check{f}^\mu x^\nu g_{\mu\nu}$

equal for all $x^\nu \Rightarrow f_\mu = g_{\mu\nu} \check{f}^\nu$

$x \to (y, Ax)$ is a linear map

then $\exists\ z \in V$ such that $(z, x) = (y, Ax)$

Hermitian adjoint $z = A^+ y$

$\Rightarrow (A^+ y, x) = (y, Ax)$

<u>matrix representation</u>

$x = e_\mu \qquad y = e_\nu$

then $(y, Ax) = (e_\nu, A e_\mu) = (e_\nu, a_{\mu\rho} e_\rho)$

$= a_{\mu\rho}\, g_{\nu\rho}$

$(A^+ y, x) = (A^+ e_\nu, e_\mu) = (\hat{a}_{\nu\rho} e_\rho, e_\mu)$

$= \hat{a}^*_{\nu\rho}\, g_{\rho\mu}$

orthonormal basis $g_{\mu\nu} = \delta_{\mu\nu}$

$\Rightarrow \qquad \hat{a}^*_{\nu\mu} = a_{\mu\nu}$

$\Rightarrow \quad \hat{a} = a^+$

Direct sum of vector spaces

U, V vector spaces

$$U \oplus V = \{ \lambda (\underset{\Uparrow \atop U}{u_1}, \underset{\Uparrow \atop V}{v_1}) + \mu (\underset{\Uparrow \atop U}{u_2}, \underset{\Uparrow \atop V}{v_2})$$

Quotient space

$U \subset W$ V is called complementary space if $W = U \oplus V$

$U \cap V = 0$

V is not unique

eg: $W = \mathbb{R}^3$ $V = \lambda(\alpha, \beta, \gamma)$

$U = \mathbb{R}^2$ $\gamma \neq 0$

$\forall \alpha, \beta, \gamma$

Quotient spaces are unique

W/U $x \in W/U$

$y \in W/U$

then $x \sim y$ if $x - y \in U$

equivalence relation.

W/U is a coset and the set of equivalence elements is called an equivalence class

$A: U \to V$

co kernel $V/\operatorname{Im} A$

orthogonal complement

$$U^{\perp} = \{x \in W \mid (x, y) = 0 \ \forall y \in U$$

$$W = U \oplus U^{\perp}$$

$$\dim W/U = \dim U^{\perp} = \dim W - \dim U$$

<u>Projection operator</u>

P ; $P^2 = P$ $P: U \to V$

if $x \in \operatorname{Im} P$ then $\exists y \mid x = Py$.

then $Px = P^2 y = Py = x$

$\Rightarrow$ if $P(x) = 0$ then $x = 0$

$\Rightarrow \{0\} = \ker P \cap \operatorname{Im} P$

$\Rightarrow \quad V = \ker P \oplus \operatorname{Im} P$ (2)

If $x \in \operatorname{Im} P$
$\Rightarrow \exists y \mid x = Py$
$Px = P^2 y = Py = x$
so $Px = 0 \Rightarrow x = 0$

so only $x = 0$
is in $\operatorname{Ker} P \cap \operatorname{Im} P$
$\Rightarrow V = \ker P \oplus \operatorname{Im} P$

Linear equations

$Ay = b \qquad A$ is $m \times n$ matrix

$m < n$ solution is not unique
$m > n$ solution may not exist.

rank of a square matrix

$\dim \operatorname{Ker} A = \dim \ker A^{+}$

column rank and row rank of a square matrix are the same

(R5)

proof

$x \in \ker A \Rightarrow (y, Ax) = 0 \quad \forall y$

$$\overset{\shortparallel}{(A^+ y, x)}$$

$\Rightarrow x \perp \operatorname{Im} A^+$

$\Rightarrow (x, A^+ y) = 0 \quad \forall y \in V$

$\Rightarrow (Ax, y) = 0 \quad \forall y \in V$

$\Rightarrow (x, A^+ y) = 0 \quad \forall y \in V$

$\Rightarrow x \in (\operatorname{Im} A^+)^\perp$

$\Rightarrow \ker A = (\operatorname{Im} A^+)^\perp$

Start with A^+ then $\ker A = (\operatorname{Im} A)^\perp$

we have that $\dim \ker A + \dim \operatorname{Im} A = \dim V$
$\dim \ker A^+ + \dim \operatorname{Im} A^+ = \dim V$

$\Rightarrow \dim \ker A = \dim (\operatorname{Im} A^+)^\perp$

$= \dim V - \dim \operatorname{Im} A^T$

$= \dim V - (\dim V - \dim \ker A^+)$

$= \dim \ker A^+$

Fredholm alternative

I) Either $Ax = b$ has a unique solution or $Ax = 0$ has a nontrivial solution

proof if $Ax_0 = 0$ $x_0 \neq 0$

then also $A(x + x_0) = b$
$$\overset{\shortparallel}{A(x)}$$

if $\exists b \in V$ such that $b \notin \operatorname{Im} A$

then $b \in \operatorname{Im} A^{\perp}$ and $\dim \operatorname{Im} A^{\perp} = \dim \ker A$

$\Rightarrow \ker A \neq \phi \Rightarrow Ax = 0$ has a nontrivial solution.

II) If $Ax = 0$ has n independent solution then also $A^T x = 0$ has

proof $\dim \ker A = \dim \ker A^T$

III) If II) holds then $Ax = b$ has no solutions unless

$b \perp$ all solutions of $A^T x = 0$

proof if b // solution of $A^T x = 0$

then $A^T b = 0$

$\Rightarrow b \in \ker A^T = \operatorname{Im} A^{\perp} \Rightarrow$

$Ax = b$ has no solutions

Determinants

$$(A)_{kl} = a_{kl} \qquad n \times n \text{ matrix}$$

$$\det A = \varepsilon_{i_1 \dots i_n} a_{1 i_1} \cdots a_{n i_n}$$

$\varepsilon_{1,2,\dots n} = 1$ is totally anti-symmetric

$$\varepsilon_{1234} = -\varepsilon_{2134} = -\varepsilon_{1324} = \varepsilon_{2341}$$

etc.

- $\det \lambda A = \lambda^n \det A$
- $\det A$ changes sign under the interchange of two rows
- det is linear in each row.

$$\det \begin{vmatrix} \lambda a_{11} + \mu b_{11} + & \cdots & + \lambda a_{1n} + \mu b_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & \cdots a_{nn} \end{vmatrix}$$

$$= \lambda \begin{vmatrix} a_{11} & \cdots a_{1n} \\ \vdots & \vdots \\ a_{n1} & \cdots a_{nn} \end{vmatrix} + \mu \begin{vmatrix} b_{11} & \cdots b_{1n} \\ a_{21} & \cdots a_{2n} \\ \vdots & \vdots \\ a_{n1} & \cdots a_{nn} \end{vmatrix}$$

- The determinant is a skew-symmetric n-linear form

$$\omega : V \times V \times \cdots \times V \longrightarrow \mathbb{F}$$

This space is denoted by $\wedge^n (V^*)$

$$\omega(\lambda a + \mu b, c_2 \ldots, c_n) = \lambda \omega(a, c_2, \ldots c_n) + \mu \omega(b, c_2 \ldots c_n)$$

$$\omega(\ldots a \ldots b \ldots) = -\omega(\ldots b \ldots a \ldots)$$

<u>example</u>: $\quad a_i = a_{ij} e_j$

$$\omega(a_1 \ldots a_n) = a_{1 i_1} \ldots a_{n i_n} \omega(e_{i_1} \ldots e_{i_n})$$

$$= \underbrace{a_{1 i_1} \ldots a_{n i_n} \varepsilon_{i_1 \ldots i_n}}_{\det A} \omega(e_1 \ldots e_n)$$

<u>Definition of ω_A</u>

$$\omega_A (x_1 \ldots x_n) = \omega(A x_1 \ldots A x_n)$$

$$= \det A \, \omega(x_1 \ldots x_n)$$

we only have to show this for a basis $e_1 \ldots e_n$ $\quad A e_k = A_{k\ell} e_\ell$ which we did in the example we can use this to prove that

$\det AB = \det A \det B$

$$\omega_{AB}(x_u) = \omega(AB\, x_u)$$

$$= \det A\, \omega(B\, x_u)$$

$$= \det A \det B\, \omega\, x_u)$$

$\|$

$\det AB\, \omega(x_u)$

$\det A^T = \det A$

proof: $\det A = \sum_\pi \varepsilon_{\pi(1)\cdots\pi(n)} A_{1\pi(1)} \cdots A_{n\pi(n)}$

$$= \sum_\pi \varepsilon_{\pi(1)\cdots\pi(n)} A_{\pi^{-1}(1)1} \cdots A_{\pi^{-1}(n)n}$$

$$\varepsilon_{\pi(1)\cdots\pi(n)} = \varepsilon_{\pi^{-1}(1)} \cdots \varepsilon_{\pi^{-1}(n)}$$

$$= \sum_\pi \varepsilon_{\pi^{-1}(1)\cdots\pi^{-1}(n)} A_{\pi^{-1}(1)1} \cdots A_{\pi^{-1}(n)n}$$

$$= \det A^T$$

$$= \sum_{\pi^{-1}}$$

note that $\varepsilon_{\pi(1)\cdots\pi(n)} = sg\,\pi$

$$\varepsilon_{\pi\pi^{-1}(1)\cdots\pi\pi^{-1}(n)} = sg(\pi\pi^{-1}) = 1$$

$$= sign\,\pi\ \varepsilon_{\pi^{-1}(1)\cdots\pi^{-1}n} = sg\,\pi\, sg\,\pi^{-1}$$

Adjugate matrix

$$A = \{ a_{k\ell} \}$$

cofactor $\qquad c_{ij} = (-1)^{i+j} \, M_{ij}$

determinant of matrix with row i and column j deleted

$$\sum a_{ij} \, C_{i'j} = \delta_{ii'} \det A$$

- Laplace expansion of determinant
- Follows from the definition of a determinant

Adjugate matrix $(\mathrm{Adj}\, A)_{ij} = C_{ji}$

$$\Rightarrow \quad A \cdot \mathrm{Adj}(A) = \det A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{\det A} \, \mathrm{Adj}\, A$$

Characteristic equation

$$\det(A - \lambda \mathbb{1}) = 0$$

λ ← eigenvalues of A

$$(-1)^n(\lambda^n - \lambda^{n-1}\,\mathrm{Tr}\,A + \cdots + (-1)^n \det A)$$

Cayley's theorem: A satisfies this equation

proof $\det(A - \lambda I)\mathbb{1} = (A - \lambda I)\,\mathrm{Adj}(A - \lambda I)$

$$= (A - \lambda I)(c_0 \lambda^{n-1} + \cdots + c_{n-1})$$

← matrices →

The rhs vanishes for $\lambda = A$ qed

see 31a)

Derivative of a determinant

$$\det A = \varepsilon_{i_1 \cdots i_n} a_{1i_1}(x) \cdots a_{ni_n}(x)$$

functions of x

$$\frac{d}{dx} \det A = \varepsilon_{i_1 \cdots i_n} \frac{da_{1i_1}}{dx} a_{2i_2} \cdots a_{ni_n}$$

$$+ \cdots$$

$$+ \cdots \varepsilon_{i_1 \cdots i_n} a_{1i_1} a_{2i_2} \cdots \frac{d}{dx} a_{ni_n}$$

$$\det(A-\lambda I) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0)$$

$$(A-\lambda I)(C_0\lambda^{n-1} + \cdots + C_{n-1})$$

$A^n \quad (-C_0 = (-1)^n)$

$A^{n-1} \times (A C_0 - C_1 = a_{n-1}(-1)^n)$

$A^{n-2} \times (A C_1 - C_2 = a_{n-1}(-1)^n)$

$\vdots$

$A \times (A C_{n-2} - C_{n-1} = (-1)^n a_{n-1})$

$A^0 \times (A C_{n-1} = (-1)^n a_0)$

$+$

$$0 = (-1)^n A^n + a_{n-1}(-1)^n A^{n-1} + \cdots + (-1)^n a_0$$

(D c)

Let us look at

$$\varepsilon_{i_1 \cdots i_n} a_{1 i_1} a_{2 i_2} \frac{d}{dx} a_{3 i_3} \cdots a_{n i_n}$$

$$= \frac{d}{dx} a_{3 i_3} \, \varepsilon_{i_1 i_2 i_3 \cdots i_n} \, a_{1 i_1} a_{2 i_2} a_{4 i_4} \cdots a_{n i_n}$$

$$\parallel$$

$$\varepsilon_{i_3 \underbrace{i_4 i_5 i_6 \cdots i_n i_1 i_2}}$$

$$\parallel$$

$$(-1)^{3+i_3} \, \varepsilon_{i_4 i_5 \cdots i_n i_1 i_3}$$

completely anti-symmetric det of A with rows and column is removed

$$\Rightarrow \frac{d}{dx} \det A = \frac{d}{dx} a_{ij} \, C_{ij}$$

$$= \frac{d}{dx} a_{ij} \left(A^{-1}\right)_{ji} \det A$$

$$\Rightarrow \frac{1}{\det A} \frac{d}{dx} \det A = \mathrm{Tr}\left(\frac{dA}{dx} A^{-1}\right)$$

$$\parallel$$

$$\frac{d}{dx} \log \det A$$

if $x = a_{ij}$ then $\dfrac{d}{da_{ij}} \log \det A = A^{-1}_{ji}$

Diagonalization

A is a linear map

$$A x = \lambda x$$

λ eigenvalue, x eigenvector, $x \neq 0$

- A is Hermitian $A^+ = A$

Then all eigenvalues are real

Proof: $\lambda (x, x) = (x, \lambda x) = (x, A x)$
$= (A^+ x, x) = (A x, x) = (\lambda x, x)$
$= \lambda^* (x, x)$

$x \neq 0 \Rightarrow \lambda = \lambda^*$

- $\lambda_k \neq \lambda_l$ then $(x_k, x_l) = 0$

$A = A^+$

proof $(x_k, A x_l) = (A x_k, x_l)$

$\Rightarrow (\lambda_l - \lambda_k)(x_k, x_l) = 0$

$\lambda_l \neq \lambda_k \Rightarrow (x_k, x_l) = 0$

Note that if $\lambda_k = \lambda_l$ eigenvectors are not necessarily orthogonal

- A operator is diagonalizable if $\exists$ basis such that

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

Not all operators are diagonalizable

Eg $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable

eigenvalues $\lambda^2 = 0$

but $V \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} V^{-1} = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let us try $\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{pmatrix} b \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow b = 0 \qquad a = 1$$
$$\lambda = 0$$

If the equation of lowest degree satisfied by has repeated roots, A is not diagonalizable.

<u>Example</u> $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$x^2 = 0$ has two roots of zero

$\Rightarrow$ A is not diagonalizable

If A is diagonalizable but has repeated roots then all roots of the minimal equation are different.

Example $A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}$

Since A satisfies the characteristic equation we have

$$(A-\lambda_1)(A-\lambda_1)(A-\lambda_2)(A-\lambda_3) = 0$$

The minimal equation is

$$(A-\lambda_1)(A-\lambda_2)(A-\lambda_3) = 0$$

and all roots are different.

Jordan canonical form

Any linear map may be brought into the Jordan canonical form with eigenvalues on the diagonal and 1 or 0 above the diagonal

$$A = \begin{pmatrix} \lambda_1 & 1 & & & & \\ & \lambda_2 & 1 & & \Theta & \\ & & \lambda_3 & 0 & & \\ & & & \lambda_4 & 0 & \\ & \Theta & & & \lambda_5 & 1 \\ & & & & & \lambda_6 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

$$\det(A - \lambda) = (\lambda_1 - \lambda)^3$$

$$A^2 = \begin{pmatrix} \lambda_1^2 & +2\lambda_1 & 1 \\ 0 & \lambda_1^2 & 2\lambda_1 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 \\ 0 & \lambda_1^3 & 3\lambda_1 \\ 0 & 0 & \lambda_1^3 \end{pmatrix}$$

$$A^3 - 3A^2\lambda_1 + 3A\lambda_1^2 + \lambda_1^3 = 0$$

lowest order eq. satisfied by A

$$A^2 + aA + b = 0$$

$$\begin{pmatrix} \lambda_1^2 + a\lambda_1 + b & 2\lambda_1 + a & 1 \\ 0 & \lambda_1^2 + a\lambda_1 + b & 2\lambda_1 + a \\ 0 & 0 & \lambda_1^2 + a\lambda_1 + b \end{pmatrix} = 0$$

has no solution

<u>If we have</u>

$$A = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

then $A^2 =$ $\begin{pmatrix} \lambda_1^2 & 2\lambda_1 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_1^2 \end{pmatrix}$

$$A^3 = \begin{pmatrix} \lambda_1^3 & 3\lambda_1^2 & 0 \\ 0 & \lambda_1^3 & 0 \\ 0 & 0 & \lambda_1^3 \end{pmatrix}$$

$$A^2 - 2\lambda_1 A + \lambda_1^2 = 0$$

$$\lambda^2 - 2\lambda_1 + \lambda_1^2 = (\lambda - \lambda_1)^2$$

has repeated roots

Quadratic Forms

$$B : V \times V \to \mathbb{R}$$

symmetric $\quad B(x,y) = B(y,x)$

bilinear $\quad B(ax_1 + bx_2, y) = aB(x_1,y) + bB(x_2,y)$

we only have to consider

$$Q(x) = B(x,x)$$

because

$$Q(x+y) = B(x+y, x+y)$$

$$= B(x,x) + B(y,y) + 2B(x,y)$$

$$\Rightarrow \quad B(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$$

$$Q(x) = x^T A x$$

A is symmetric matrix

Lagrange method of diagonalizing a quadratic form

$$Q = x^2 - y^2 - z^2 + 2xy - 4xz + 6yz$$

$+4yz \quad -z^2$

$+6yz$

$$= (x+y-2z)^2 - 2y^2 - 4z^2$$

$$= (x+y-2z)^2 - 2\left(y - \tfrac{5}{2}z\right)^2 + \tfrac{15}{2}z^2$$

$$= \xi^2 - 2\eta^2 + \tfrac{15}{2}y^2$$

$$Q = x^T A x$$

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -1 & 3 \\ -2 & 3 & -1 \end{pmatrix}$$

$$\det(A - \lambda) = \left| \begin{pmatrix} 1-\lambda & 1 & -2 \\ 1 & -1-\lambda & 3 \\ -2 & 3 & -1-\lambda \end{pmatrix} \right|$$

$$= (1-\lambda)((-1-\lambda)^2 - 9) - 1\left((-1-\lambda) + 6 \right) - 2\left(3 - 2(1+\lambda)\right) = 0$$

$$= (1-\lambda)(\lambda^2 + 2\lambda - 8) + \lambda - 5 - 2 + 4\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 10\lambda - 8 + 5\lambda - 7 = 0$$

$$-\lambda^3 - \lambda^2 + 15\lambda - 15 = 0$$

A can be diagonalized by orthogonal matrix

$$A = O^T \Lambda O$$

$$\Rightarrow Q = x^T O^T \Lambda O x$$

$$= (Ox)^T \Lambda O x$$

note that this generally give a different result than the Lagrange method because in that case the basis is not orthogonal

Symplectic form

skew-symmetric linear form

$\omega(x, y) = -\omega(y, x)$ $\qquad \omega : V \times V \to \mathbb{R}$

e_i basis $\qquad \omega(e_i, e_j) = \omega_{ij}$

$x = x_i\, e_i \qquad y = y_j\, e_j$

then $\omega(x, y) = \omega_{ij}\, x_i y_j$

By diagonalization ω can be brought into standard form

$$\omega = \sigma^T \begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & 0 & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & 0 & & & & 0 & \lambda_n \\ & & & & & -\lambda_n & 0 \end{pmatrix} \sigma$$

with σ an orthogonal transformation

$$\sigma^T \sigma = \mathbb{1}$$

use the columns of σ as basis, then

$$\omega'_{ij} = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

we can absorb the λ_k in the normalization of the basis

Formal construction of a basis

Wedge product $e^{*i} \wedge e^{*j} : V^* \times V^* \to \mathbb{R}$

$$e^{*i} \wedge e^{*j} (e_\alpha, e_\beta) = \delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha$$

$$e^{*i} \wedge e^{*j} (x^\alpha e_\alpha, x^\beta e_\beta) = x^i y^j - x^j y^i$$

$$\omega(x, y) = \frac{1}{2} \omega_{ij} (x^i y^j - y^i x^j)$$

use anti-symmetry

So we can expand

$$\omega = \frac{1}{2} \omega_{ij} e^{*i} \wedge e^{*j}$$

Construction of basis

$$\omega = \underbrace{\left(e^{*1} - \frac{1}{\omega_{12}} (\omega_{23} e^{*3} + \dots + \omega_{2n} e^{*n})\right)}_{f^{1*}} \wedge \underbrace{(\omega_{12} e^{*2} + \omega_{13} e^{*3} + \dots + \omega_{1n} e^{*n})}_{f_2^*} + \underbrace{\omega^{[3]}}_{\text{does not contains } e^{*1} \text{ and } e^{*2}}$$

$$\Rightarrow \omega = f^{1*} \wedge f^{2*} + \omega^{(3)}$$

continue recursively

f_k is basis of V dual to f^*_k

$f^*_k(f_\ell) = \delta_{k\ell}$

then $\omega(f_1, f_2) = (f^{1*} \wedge f^{2*})(f_1, f_2)$

$\quad \| $

$-\omega(f_2, f_1) = 1$

same $\omega(f_3, f_4) = -\omega(f_4, f_3) = 1$

If $f^{*i} = a^i_j\, e^{*j} \Rightarrow e^{*j} = a^{-1\,j}_{\ k} f^{*k}$

then $e_i = f_j\, a^j_{\ i}$

$f^{*k}(e_\ell) =$

$f^*_k(a^m_{\ \ell} f_m) = a^m_{\ k}$

Let us show this; Let $e_i = f_j\, b^j_{\ i}$

$\delta^j_{\ i} = e^{*j}(e_i)$

$= a^{-1\,j}_{\ k} \underbrace{f^{*k}(f_j)}_{\delta_{jk}} b^j_{\ i}$

$= a^{-1\,j}_{\ k}\; b^k_{\ i} \Rightarrow b^k_{\ i} = a^k_{\ i}$

$\Rightarrow e_i = f_j\, a^j_{\ i}$

Let $\hat{\Omega} = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & \\ & & \ddots \end{pmatrix}$

then $\omega = f^{1*} \wedge f^{2*} + f^{3*} \wedge f^{4*} + \cdots$

$$\omega(f_k, f_l) = \hat{\Omega}_{kl}$$

$$\omega(e_k, e_l) = \omega(a^m{}_k f_m, a^n{}_l f_n)$$

$$= a^m{}_k\, a^n{}_l\, \omega(f_m, f_n)$$

$$= a^m{}_k\, a^n{}_l\, \hat{\Omega}_{mn}$$

$$= a^T \hat{\Omega}\, a$$

So a brings the skew symmetric form into a standard form