Quadratic form

symmetric bilinear form $B: V \times V \to \mathbb{R}$

quadratic form $Q(x) = B(x,x)$

$$B(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right)$$

in matrix representation

$$B(x,x) = x^T M x$$

$M^T = M \Rightarrow$ it can be diagonalized by an orthogonal transformation

$$M = O^T \Lambda O$$

Λ diagonal.

$$\Rightarrow B(x,x) = x^T O^T \Lambda O x$$
$$= (Ox)^T \Lambda O x$$

Lagrange method of diagonalizing a quadratic form

$$Q = x^2 - y^2 - z^2 + 2xy - 4xz + 6yz$$
$$= (x + y - 2z)^2 - 2y^2 + 10yz - 4z^2 - z^2$$
$$= (x + y - 2z)^2 - 2\left(y - \frac{5}{2}z\right)^2 + \frac{15}{2}z^2$$
$$= \xi^2 - 2\eta^2 + \frac{15}{2}\zeta^2$$

Symplectic form

skew symmetric form $\omega : V \times V \to \mathbb{R}$

$$\omega(x,y) = -\omega(y,x)$$

basis $\omega(e_i, e_j) = \omega_{ij}$

$$x = x^i e_i \quad y = y^i e_i$$

then $\omega(x,y) = \omega_{ij} x^i y^j$

$$\omega(x,y) = x^T \Omega y$$

Since Ω is antisymmetric it cannot be diagonalized

wedge product $e^{*i} \wedge e^{*j} : V^* \times V^* \to \mathbb{R}$

$$e^{*i} \wedge e^{*j} (e_\alpha, e_\beta) = \delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha$$

$$e^{*i} \wedge e^{*j} (x^\alpha e_\alpha, y^\beta e_\beta) = (\delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha) x^\alpha y^\beta$$
$$= x^i y^j - x^j y^i$$

$$\Rightarrow \quad \omega(x,y) = \frac{1}{2} \omega_{ij} (x^i y^j - y^i x^j)$$
$$= \omega_{ij} \, x_i x_j$$

So we can expand $\omega = \frac{1}{2} \omega_{ij} e^{*i} \wedge e^{*j}$

Construction of basis

$$\omega = \tfrac{1}{2}\omega_{ij}\, e^{*i} \wedge e^{*j}$$

$$\omega = \underbrace{\left(e^{*1} - \frac{1}{\omega_{12}}\left(\omega_{23}\, e^{*3} + \dots \omega_{2n}\, e^{*n}\right)\right)}_{f^{1*}} \wedge \Big($$

$$\underbrace{\left(\omega_{12}\, e^{*2} + \omega_{13}\, e^{*3} + \dots + \omega_{1n}\, e^{*n}\right)}_{f^{2*}}$$

$$+ \, \omega^{[3]}$$

← does not contain e_1^* or e_2^*

$$\Rightarrow \omega = f^{1*} \wedge f^{2*} + \omega^{[3]}$$

continue recursively

f is basis of V dual to f^*

then $\omega(f_1, f_2) = -\omega(f_2, f_1)$

$= \omega(f_3, f_4) = -\omega(f_4, f_3) = 1$

$f^{*i} = a^i{}_j\, e^{*j} \Rightarrow e_i = f_j\, a^j{}_i$

$\Rightarrow \Omega = A^T \cdot \tilde{\Omega}\, A$

$$\tilde{\Omega} = \begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & & \ddots \end{pmatrix}$$