Determinants

$A = a_{ij}$

$\det A = \varepsilon_{i_1 \dots i_n} a_{1 i_1} \cdots a_{n i_n}$

$\varepsilon_{12 \dots n} = 1$ and it totally antisymmetric.

- $\det \lambda A = \lambda^n A$

- $\det A$ changes sign under the interchange of two rows

- determinant is linear in each row

$$\det \begin{vmatrix} \lambda a_{11} + \mu b_{11} & \dots & \lambda a_{1n} + \mu a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{vmatrix}$$

$$= \lambda \det \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} + \mu \det \begin{pmatrix} b_{11} & \dots & b_{1n} \\ a_{21} & \dots & a_{21} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

- the determinant is a skew symmetric n-linear form

$\omega : V \times V \times \dots \times V \to \mathbb{F}$

This space is denoted by $\Lambda^n(V^*)$

$\omega(\lambda a + \mu b, c_2 \dots c_n) = \lambda \omega(a, c_2 \dots c_n) + \mu \omega(b, c_2 \dots c_n)$

$\omega(\dots \underset{i}{a} \dots \underset{j}{b} \dots) = -\omega(\dots \underset{i}{b} \dots \underset{j}{a} \dots)$

example $a_i = a_{ij} e_j$

$$\omega(a_1 \dots a_n) = a_{1i_1} \cdots a_{ni_n}\, \omega(e_{i_1}, \dots, e_{i_n})$$

$$= \underbrace{a_{1i_1} \cdots a_{ni_n} \varepsilon_{i_1 \cdots i_n}}_{\det A}\, \omega(e_1 \dots e_n)$$

Definition of ω_A

$$\omega_A(x_1 \dots x_n) = \omega(\underbrace{A_1 x_1}_{A_{1k} x_k} \cdots A x_n)$$

$$= \text{"} \det A\, \omega(x_1 \dots x_n)$$

proportionality constant is 1 (go to basis).

Can beused the prove properties of determinants.

$$\omega_{AB}(x_k) = \omega(AB\, x_k) = \det AB\, \omega(x_k)$$

$$= \det A\, \omega(B\, x_k) = \det A \det B\, \omega(x_k)$$

$$\Rightarrow \det AB = \det A \det B$$

We also have $\det A^T = \det A$

$$\det A = \sum_{\pi} \varepsilon_{\pi(1)\cdots\pi(n)} A_{1\,\pi(1)} \cdots A_{n\,\pi(n)}$$

$$= \sum_{\pi} \varepsilon_{\pi(1)\cdots\pi(n)} A_{\pi^{-1}(1)\,1} \cdots A_{\pi^{-1}(n)\,n}$$

$$\varepsilon_{\pi(1)\cdots\pi(n)} = \varepsilon_{\pi^{-1}(1)\cdots\pi^{-1}(n)}$$

$$= \sum_{\pi} \varepsilon_{\pi^{-1}(1)\cdots\pi^{-1}(n)} A_{\pi^{-1}(1)\,1} \cdots A_{\pi^{-1}(n)\,n}$$

$$\sum_{\pi} = \sum_{\pi^{-1}}$$

$$= \det A^T$$

Adjugate matrix

$$A = \{a_{kc}\}$$

cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ determinant of matrix with row and column of a_{ij} deleted.

$$\sum_j a_{ij} C_{i'j} = \delta_{ii'} \det A$$

Laplace expansion of determinant
- follows from definition of determinant

Adjugate matrix $(\mathrm{Adj}\,A)_{ij} = C_{ji}$

then $A \,\mathrm{Adj}(A) = \det A$

$$\Rightarrow A^{-1} = \frac{1}{\det A} \operatorname{Adj} A$$

Cayley theorem

characteristi equation

$\det(A - \lambda \mathbb{1}) = 0$

λ are the eigenvalues of A

$$\det(A - \lambda \mathbb{1}) = (-1)^n \left(\lambda^n - \lambda^{n-1} \operatorname{tr} A + \dots + (-1)^n \det A \right)$$

Cayley's theorem: A satisfies this equation

proof $\det(A - \lambda I) = (A - \lambda I) \operatorname{Adj}(A - \lambda I)$

$$(-1)^n (\lambda^n + \alpha_1 \lambda^{n-1} + \dots + \alpha_n) = (A - \lambda I)(c_0 \lambda^{n-1} + \dots + c_{n-1})$$

the rhs is zero for $\lambda = A$

ϵ.

Derivative of a determinant

$$\det A = \sum_{i_1 \ldots i_n} a_{1 i_1} \cdots a_{n i_n}$$

functions of x

$$\frac{d}{dx}(\det A) = \sum_{i_1 \ldots i_n} \big(a'_{1 i_1} a_{2 i_2} \cdots a_{n i_n}$$
$$+ a_{1 i_1} a'_{2 i_2} \cdots a_{n i_n}$$
$$+ a_{1 i_1} a_{2 i_2} \cdots a'_{n i_n} \big)$$

some of n determinant

use Laplace expansion

$$= a'_{ij} C_{ij} \quad \leftarrow \text{cofactor}$$

$$= a'_{ij} (A^{-1})_{ji} \det A$$

$$\Rightarrow \frac{1}{\det A} \frac{d}{dx} \det A = \operatorname{Tr} \frac{dA}{dx} A^{-1}$$

$$\parallel$$

$$\frac{d}{dx} \log \det A$$

if $x = a_{ij}$ then $\frac{d}{d a_{ij}} \log \det A = A^{-1}_{ji}$

Diagonalization

A is linear map

$$A x = \lambda x$$

eigenvalue eigenvector

- Hermitian $A = A^{+}$
then all eigenvalues are real

$$\lambda (x,x) = (x, \lambda x) = (x, Ax)$$
$$= (A^{+}x, x)$$
$$= (Ax, x)$$
$$= (\lambda x, x)$$
$$= \lambda^{*} (x,x)$$

$$\Rightarrow \lambda^{*} = \lambda$$

- $\lambda_k \neq \lambda_e$ then $(x_k, x_e) = 0$
$A = A^{+}$

$$(x_k\ A x_e) = (A x_k, x_e)$$
$$\lambda_e (x_k, x_e) = \lambda_k (x_k, x_e)$$
$$\lambda_k \neq \lambda_e \rightarrow (x_k, x_e) = 0$$

- An operator is diagonalizable if $\exists$ basis for which

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

not all matrices are diagonalizable.

If the equation of lowest degree satisfied by A has repeated roots A is not diagonizable.

Reason: If A is diagonalizable

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{pmatrix}$$

then minimal equation is $(A-\lambda_1)(A-\lambda_2) \times (A-\lambda_3) = 0$

so all roots are different.

Jordan canonical form

Any linear map may be brought into the Jordan canonical form with eigenvalues on the diagonal and 1 or 0 above the diagonal

Example: $A = \begin{pmatrix} \lambda_1 & 1 & & & & \\ & \lambda_1 & 1 & & \Theta & \\ & & \lambda_1 & 0 & & \\ & \Theta & & \lambda_2 & 1 & \\ & & & & \lambda_2 & 0 \\ & & & & & \lambda_2 \end{pmatrix}$

$$\det(A-\lambda) = (\lambda-\lambda_1)^3(\lambda-\lambda_2)^3$$

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 & 0 \\ & \lambda_1 & 1 \\ \Theta & & \lambda \end{pmatrix} = \begin{matrix} \lambda_1^2 & 2\lambda_1 & 1 \\ 0 & \lambda_1^2 & 2\lambda_1 \\ 0 & 0 & \lambda_1^2 \end{matrix}$$

$$\begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}^3 = \begin{pmatrix} \lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 \\ 0 & \lambda_1^3 & 3\lambda_1 \\ 0 & 0 & \lambda_1^3 \end{pmatrix}$$

$$\lambda^3 - 3\lambda^2\lambda_1 + 3\lambda\lambda_1^2 - \lambda_1^3$$

$$+\begin{pmatrix} -3\lambda_1^3 & -6\lambda_1^2 & -3\lambda_1 \\ & -3\lambda_1^3 & -6\lambda_1^2 \\ & & -3\lambda_1^3 \end{pmatrix} \qquad \begin{pmatrix} 3\lambda_1^3 & 3\lambda_1^2 & 0 \\ & 3\lambda_1^3 & 3\lambda_2 \\ & & 3\lambda_2^3 \end{pmatrix}$$

$$\begin{matrix} \lambda_1^3 & 3\lambda_1^2 & 3\lambda_1 \\ & \lambda_1^3 & 3\lambda_2 \\ & & \lambda_1^3 \end{matrix}$$

$$\begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{pmatrix}^2 = \begin{pmatrix} \lambda_2^2 & 2\lambda_2 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\lambda^2 - 2\lambda_2\lambda + \lambda_2^2$$

but for

$$\begin{matrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{matrix}$$

$$\lambda_2^2 - 2\lambda_2^2 + \lambda_2^2 = 0$$

$\Rightarrow$ minimal characteristic polynomial is of second order.

$$\begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{pmatrix}^2 = \begin{pmatrix} \lambda_2^2 & 2\lambda_2 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\lambda^2 - 2\lambda_2\lambda + \lambda_2^2$$

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