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Euler angles

Euler angle parameterization of $SU(2)$

$$u = e^{-i\phi\sigma_3/2}\, e^{-i\theta\sigma_2/2}\, e^{-i\psi\cdot\sigma_3 2}$$

$$= \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi_2} \end{pmatrix} \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ +\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{-\frac{i\psi}{2}} & \\ & e^{i\psi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{i}{2}(\phi+\psi)}\cos\frac{\theta}{2} & -e^{i(\psi-\phi)/2}\sin\frac{\theta}{2} \\ e^{i(\phi-\psi)}\sin\frac{\theta}{2} & e^{\frac{i}{2}(\phi+\psi)}\cos\frac{\theta}{2} \end{pmatrix}$$

the ϕ, θ, ψ are the euler angls

$$x_0 = \cos\frac{\theta}{2}\cos((\phi+\psi)/2)$$

$$x_1 = \sin\frac{\theta}{2}\sin((\phi-\psi)/2)$$

$$x_2 = -\sin\frac{\theta}{2}\cos((\phi-\psi)/2)$$

$$x_3 = -\cos\frac{\theta}{2}\sin((\phi+\psi)/2)$$

Volume element of $SU(2)$

$$ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$

$$dx_0 = d\left(\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}\right)$$

$$= -\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\phi+\psi}{2}\,d\theta - \frac{1}{2}\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}\,d(\psi+\phi)$$

$$dx_3 = \frac{1}{2}\sin\frac{\theta}{2}\sin\frac{\phi+\psi}{2}\,d\theta - \frac{1}{2}\cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}\,d(\psi+\phi)$$

$$dx_1 = \frac{1}{2}\cos\theta/2\,\sin\left(\frac{\phi-\psi}{2}\right)d\theta + \frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\phi-\psi}{2}\,d(\phi-\psi)$$

$$dx_2 = -\frac{1}{2}\cos(\phi-\psi)/2\,d\theta + \frac{1}{2}\sin\frac{\theta}{2}\sin\frac{\phi-\psi}{2}\,d(\phi-\psi)$$

$$dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$
$$= \frac{1}{4}d\theta^2 + \frac{1}{4}\cos^2\frac{\theta}{2}\left(d(\psi+\phi)\right)^2 + \frac{1}{4}\sin^2\frac{\theta}{2}\,d^2(\phi-\psi)$$

$$\frac{1}{4}\begin{pmatrix} d\theta \\ d\phi \\ d\psi \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \cos\theta \\ 0 & \cos\theta & 1 \end{pmatrix}}_{g_{\mu\nu}} \begin{pmatrix} d\theta \\ d\phi \\ d\psi \end{pmatrix}$$

The volume element is

$$\sqrt{\det g}\, d\theta\, d\phi\, d\psi = \frac{1}{8} \sin\theta\, d\theta\, d\phi\, d\psi$$

$\Rightarrow$ volume of $SU(2)$

$$V = \frac{1}{8}\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi = 2\pi^2$$

volume of surface of 4d sphere

$$\left.\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\right|_{d=4} = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$$

Connection between $SU(2)$ and $SO(3)$

spinor $\quad \chi \to \underbrace{D(R)}_{\text{representation of rotation group}} \chi$

Pauli matrices $\quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$

$\sigma_i' = U \sigma_i U^{-1}$ is new set of Pauli matrices satisfying the same anti-commutation relations

$\sigma_i' = R_{ij}\, \sigma_j$

basis of Hermitian traceless 2x2 matrices

$$\sigma_i' \sigma_j' + \sigma_j' \sigma_i' = 2\delta_{ij}$$

$$\sigma_k R_{ki} \sigma_l R_{lj} + \sigma_l R_{lj} \sigma_k R_{ki}$$

$$= 2\delta_{kl} R_{ki} R_{lj}$$

$$\Rightarrow \quad R_{ki} R_{kj} = \delta_{ij}$$

$$\Rightarrow \quad R^T R = 1$$

$$\Rightarrow \quad R \in O(3)$$

$SU(2)$ is connected $\Rightarrow$
R must be connected
since $\det \mathbb{1} = 1 \Rightarrow \det R = 1$
$\det R = -1$ would be disconnected.

The reverse is also true. For every rotation R ther is a unitary matrix such that $U(R)\,\sigma_i\,U^{-1}(R) = \sigma_j R_{ji}$
Also $-U(R)$ satisfies this relation.
So to each R we have two corresponding $SU(2)$ matrices

So $SO(3)$ is $SU(2)$ with the antipodal points identified

closed loop in $SO(3)$ but not contractable

contractable

$\Rightarrow$ homotopy group of $SO(3) = \mathbb{Z}_2$

$$SO(3) = SU(2)/\mathbb{Z}_2$$

Spinor representations of SO(N)

γ matrices γ_μ. They form a Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \mu = 1, \dots, 2n$$

γ_μ are $2^n \times 2^n$ matrices

(for $2n+1$ we define $\gamma_{2n+1} = -(i)^n \gamma_1 \gamma_2 \cdots \gamma_{2n}$)

A rotation by θ in the $\mu\nu$ plane is given by

$$e^{-i \frac{1}{4i}[\gamma_\mu, \gamma_\nu]\theta} \gamma_i \, e^{i\frac{1}{4i}[\gamma_\mu, \gamma_\nu]\theta} = \gamma_j R_{ji}$$

$\Rightarrow \quad \Gamma_{\mu\nu} = \frac{1}{4i}[\gamma_\mu, \gamma_\nu]$ is a basis of the Lie algebra of SO(N)

$e^{i\Gamma_{\mu\nu}\theta_{\mu\nu}}$ is called spin(N)

For $2n$

$$\Gamma_5 = \gamma_1 \cdots \gamma_{2n}$$

$$\{\Gamma_5, \gamma_\mu\} = 0$$

$$\Rightarrow [\Gamma_5, \Gamma_{\mu\nu}] = 0 \quad \forall_{\mu\nu}$$

$\Rightarrow \Gamma_5 = \lambda \mathbb{1}$ by Schur which is not possible because $\{\Gamma_5, \gamma_\mu\} = 0$

$\Rightarrow$ space is reducible according to the eigenvalues of Γ_5.

infinitesimal

$$(1 - i\Gamma_{\mu\nu}\theta)\gamma_i(1 + i\Gamma_{\mu\nu}\theta) = \gamma_j R_{ji}$$

$$\underset{1 + i\lambda^R_{ji}\theta}{\shortparallel}$$

$$\Rightarrow \gamma_j \lambda^R_{ji} = -\Gamma_{\mu\nu}\gamma_i + \gamma_i\Gamma_{\mu\nu}$$

$\cancel{\gamma}i \neq \mu, \nu \quad \Rightarrow [\gamma_i, \Gamma_{\mu\nu}] \Rightarrow$

$i = \mu \qquad\qquad \mu \neq \nu$

$$\frac{1}{4i}(\gamma_\mu\Gamma_{\mu\nu} - \Gamma_{\mu\nu}\gamma_\mu]$$

$$= \frac{1}{4i}(\gamma_\mu(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\gamma_\mu$$

$$= \frac{1}{i}\gamma_\nu$$

$i = \nu \qquad \frac{1}{4i}[\gamma_\nu\Gamma_{\mu\nu} - \Gamma_{\mu\nu}\gamma_\nu]$

$$= \frac{1}{4i}[\gamma_\nu(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\gamma_\nu)$$

$$= -\frac{1}{i}\gamma_\mu$$

$$\Rightarrow -\Gamma_{\mu\nu}\gamma_i + \gamma_i\Gamma_{\mu\nu}$$

$$= \frac{1}{i}\delta_{i\mu}\gamma_\nu - \frac{1}{i}\delta_{i\nu}\gamma_\mu$$

$$= i\,\delta_{i\nu}\gamma_\mu - i\,\delta_{i\mu}\gamma_\nu$$

$$\Rightarrow \qquad \lambda^{R\mu\nu}_{ji} = \delta_{j\mu}\delta_{i\nu} - \delta_{j\nu}\delta_{i\mu}$$

Adjoint representation

For $SU(2)$ $\quad u \sigma_i u^{-1} = \sigma_j R_{ji}$

in general
$$g(1 + i\varepsilon \lambda_i) g^{-1}$$
$$= 1 + i\varepsilon \underbrace{g \lambda_i g^{-1}}_{\sum a_{ij} \lambda_j}$$

This is called the adjoint representation

$$Ad(g) \lambda_i = g \lambda_i g^{-1} = \lambda_j (Ad(g))^j{}_i$$

Peter - Weyl theorem

Left invariant measure $\quad d\, g_1 g = dg$

Right invariant measure $\quad dg\, g_1 = dg$

For compact groups the left invariant measure is equal to the right invariant measure. It is called the Haar measure

Then $\int dg = \int d\, g_1 g$

the same way as for finite groups

just $\sum_g \longrightarrow \int dg$

All theorems for finite groups are valid

$$\frac{1}{|G|} \sum_{g \in G} D_{ij}^{\gamma}(g^{-1}) D_{kl}^{\kappa}(g) = \frac{1}{\dim \gamma} \delta_{jk} \delta_{il} \delta^{\gamma\kappa}$$

for compact groups

$$\int dg \; D_{ij}^{\gamma}(g^{-1}) D_{kl}^{\kappa}(g) = \frac{1}{\dim \gamma} \delta^{\gamma\kappa} \delta_{jk} \delta_{il}$$

$D_{ij}^{\gamma}(g)$ are functions on a group.
There cannot be more functions on a group than the number of group element for finite groups

$$\sum_{\gamma} (\dim \gamma)^2 \leq |G|$$

Actually the equal sign holds

$$\sum_{\gamma} (\dim \gamma)^2 = |G|$$

$\Rightarrow$ $D^{\gamma}(g)$ is a complete set of functions on G

<u>Peter-Weyl theorem</u> $D_{mn}^{\gamma}(g)$ are a complete set of functions on the group manifold

Representations of $SU(2)$

$$e^{-iJ_3\phi}\, e^{-iJ_2\theta}\, e^{-iJ_3\psi} = e^{-im\phi}\, d^J_{mn}(\theta)\, e^{-in\psi}$$

for spin J

$$D^J_{mn}(\theta,\psi) = \langle Jm | e^{-iJ_3\phi} e^{-iJ_2\theta} e^{-iJ_3\psi} | Jn\rangle$$

J_K are $2J+1$ dimensional

$$\frac{1}{|G|}\sum_g \to \frac{1}{vol(G)}\int dG$$

$$= \frac{1}{2\cdot 8\pi^2}\int \sin\theta\, d\theta\, d\phi\, d\psi$$

$\Rightarrow$ Orthogonality relation

$$\frac{1}{16\pi^2}\int_0^\pi \sin\theta \int_0^{2\pi} d\phi \int_0^{4\pi} d\psi\, D^{J*}_{mn}(\theta,\phi\psi)\, D^{J'}_{m'n'}(\theta,\phi\psi)$$

$$= \frac{1}{2J+1}\,\delta^{JJ'}\,\delta_{mm'}\,\delta_{nn'}$$

D^L_{m0} do not depend on ψ

integral over ψ gives 4π

$$\Rightarrow \frac{1}{4\pi}\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, D^{L*}_{m0}(\theta,\varphi)\, D^{L'}_{m'0}(\theta,\varphi)$$

$$= \frac{\delta^{LL'}}{2L+1}\,\delta_{mm'}$$

$$\Rightarrow Y^l_m(\theta,\varphi) = \sqrt{\frac{2L+1}{4\pi}}\left(D^L_{m0}(\theta,\varphi,\ldots)\right)^*$$

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The character is given by

$$\chi^j(g) = \mathrm{Tr}\, D^j(g)$$

We can calculate the trace in a basis the rotion is a rotation of ϕ about the z axis

$$\langle j n | e^{-i\phi J_3} | j n \rangle = e^{-in\phi}$$

$$\Rightarrow \chi(\phi) = e^{-ij\phi} + \dots + e^{ij\phi}$$

$$= e^{-ij\phi} \frac{(1 - e^{i(2j+1)\phi})}{1 - e^{i\phi}}$$

$$= e^{-ij\phi} \frac{e^{+i\frac{(2j+1)}{2}\phi} (-2i) \sin (2j+1)\phi}{(-2i)\, e^{i\phi/2} \sin \phi/2}$$

$$= \frac{\sin (2j+1)\phi/2}{\sin \phi/2}$$

Lie Algebras

g is a Lie algebra if

- Lie bracket $g \times g \to g$

$$[X_1, X_2]$$

a) skew symmetric

b) linear

$$[\lambda X + \mu Y, Z] = \lambda [X, Z] + \mu [Y, Z]$$

c) Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

- real $n \times n$ matrices with $[X, Y] = XY - YX$

is a algebra

- left invariant vector fields are Lie algebra

$$[L_i, L_j] = f_{ij}^{\ k} L_k$$

<u>ideal</u> subalgebra such that $[i, g] \subseteq i$

$$g - i = \{ X \mid X \sim X + I, I \in i \}$$

$$[X + i, Y + i] = [X, Y] + i$$

a Lie Algebra is simple if it has no nontrivial ideals

Adjoint representation

$$\mathrm{ad}(X)\, Y = [X, Y]$$

Jacobi identity

$$(\mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X))\, Z = \mathrm{ad}([X, Y])\, Z$$

$$\Rightarrow \left[\mathrm{ad}[X], \mathrm{ad}[Y]\right] = \mathrm{ad}\,[X, Y]$$

$\Rightarrow$ $\mathrm{ad}[X]$ is a representation of the algebra. It is called the adjoint representation.

$$\exp[\mathrm{ad}(tX)]\, Y$$

$$= Y + \mathrm{ad}(tX)Y + \tfrac{1}{2}\mathrm{ad}(tX)(\mathrm{ad}(tX)\, Y)$$

$$= Y + t(X, Y) + \tfrac{1}{2}\underbrace{\mathrm{ad}(tX)(t[X, Y]}_{t^2[[X, [X, Y]]]}$$

compare

$$e^{tA} B e^{-tA} = B + t[A, B] + \tfrac{1}{2}t^2[[A, [A, B]]$$

If the Killing metric is negative definite the Lie algebra is compact.

The g_{ij} are use to lower indices

$$f_{ki}^{\ \ l} g_{lj} = f_{kij}$$

f_{kij} is completely antisymmetric

By definition $[X_i, X_j] = i f_{ij}^{\ \ k} X_k$

$\Rightarrow$ anti-symmetry in ij

we also have that

$$\langle ad(X_k) X_i, X_j \rangle + \langle X_i, ad(X_k) X_j \rangle$$

$$\langle [X_k, X_i], X_j \rangle + \langle X_i, [X_k, X_j] \rangle = 0$$

$$f_{ki}^{\ \ l} g_{lj} + f_{kj}^{\ \ l} g_{li} = 0$$

$$f_{kij} + f_{kji} = 0$$

Then $f_{ijk} = -f_{jik} = f_{jki} = -f_{kji}$

g^{ij} is the inverse of the Killing metric

Killing form

inner product $\langle X, Y\rangle = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$

Jacobi identity (use that $\mathrm{ad}[X,Y] = [\mathrm{ad}X, \mathrm{ad}Y]$)

$$\langle \mathrm{ad}(X)Y, Z\rangle + \langle Y, \mathrm{ad}(X)Z\rangle = 0$$

$$\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0$$

$$\langle \exp(\mathrm{ad}(tX))Y, \exp(\mathrm{ad}(tX))Z\rangle = \langle Y, Z\rangle \quad (**)$$

correct for $t=0$

$$\frac{d}{dt}\exp(\mathrm{ad}(tX))Y = \exp(\mathrm{ad}(tX))\frac{d\,\mathrm{ad}(tX)}{dt}Y$$

$$= \exp\mathrm{ad}(tX)[X,Y]$$

$\frac{d}{dt}$ of lhs of $(**)$

$$\langle \underbrace{\exp(\mathrm{ad}\,tX)\circ [X,Y]}_{[[X,\ \exp(\mathrm{ad}(tX))],\ Y]}, \exp(\mathrm{ad}(tX))[X,Z]\rangle$$

use Jacobi identity

$$\langle \exp(\mathrm{ad}(tX))[X,Y], \exp(\mathrm{ad}(tX))Z\rangle$$

$(\mathrm{ad}X)Y$

$$+\langle \exp(\mathrm{ad}(tX))Y, \exp(\mathrm{ad}(tX))[X,Z]\rangle$$

$\mathrm{ad}X$ commutes with $\exp(\mathrm{ad}(tX))$

$\mathrm{ad}X$ ..... $\exp(\mathrm{ad}(tX))Y = [X, \exp(\mathrm{ad}(tX))Y]$

$\langle X, Y\rangle = \mathrm{tr}\,(\mathrm{ad}(X)\,\mathrm{ad}\,Y)$

$(**)$

$\langle \mathrm{ad}(X)Y, Z\rangle + \langle Y, \mathrm{ad}(X)Z\rangle$

$= \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle$

$= \mathrm{tr}\,\mathrm{ad}[X,Y]\,\mathrm{ad}\,Z + \mathrm{tr}\,\mathrm{ad}Y\,\mathrm{ad}[X,Z]$

$= \mathrm{tr}\,[\mathrm{ad}\,X, \mathrm{ad}\,Y]\,\mathrm{ad}\,Z + \mathrm{tr}\,\mathrm{ad}Y[\mathrm{ad}\,X, \mathrm{ad}\,Z]$

$= \mathrm{tr}\,\mathrm{ad}X\,\cancel{\mathrm{ad}Y\,\mathrm{ad}Z} - \mathrm{tr}\,\mathrm{ad}Y\,\cancel{\mathrm{ad}X\,\mathrm{ad}Z}$

$+ \mathrm{tr}\,\mathrm{ad}Y\,\cancel{\mathrm{ad}X\,\mathrm{ad}Z} - \cancel{\mathrm{tr}\,\mathrm{ad}X}\,\mathrm{ad}Z\,\mathrm{ad}X$

$= 0$

$\frac{d}{dt}\langle \exp(\mathrm{ad}\,tX)Y, \exp(\mathrm{ad}\,tX)Z\rangle$

$= \langle \exp(\mathrm{ad}\,tX)[X,Y], \exp(\mathrm{ad}\,tX)Z\rangle$

$+ \langle \exp(\mathrm{ad}\,tX)Y, \exp\,\mathrm{ad}\,tX[X,Z]\rangle$

$\exp(\mathrm{ad}\,tX)[X,Y] = \exp(\mathrm{ad}\,tX)\,\mathrm{ad}X\,Y$

$= \mathrm{ad}X\,\exp\,\mathrm{ad}\,tX\,Y$

$= \langle \mathrm{ad}\,X\exp(\mathrm{ad}\,tX)Y, \exp(\mathrm{ad}(tX))Z\rangle$

$+ \langle \exp(\mathrm{ad}\,tX)Y, \mathrm{ad}\,X\exp(\mathrm{ad}\,tX)Z\rangle$

$= 0$ using the identity for $(**)$

Killing metric tensor

$$g_{ij} = \mathrm{Tr}\ \mathrm{ad}\, X_i\, \mathrm{ad}\, X_j$$

with iX_i the generator of the group

$$(\mathrm{ad}\, X_i)\, X_j = X_k\, (\mathrm{ad}(X_i))^k{}_j$$

$$\parallel$$

$$[X_i, X_j] = i f_{ij}{}^k X_k$$

$$\Rightarrow (\mathrm{ad}\, X_i)^k{}_j = i f_{ij}{}^k$$

Now we can calculate $\mathrm{Tr}\ \mathrm{ad}\, X_i\, \mathrm{ad}\, X_j$

$$= \sum_{kl} (\mathrm{a}\, X_i)^k{}_l\ (\mathrm{ad}\, X_j)^l{}_{m=k}$$

$$= i f_{il}{}^k\ i f_{jk}{}^l$$

$$\Rightarrow g_{ij} = -f_{il}{}^k f_{jk}{}^l$$

Theorem

The algebra is semisimple if g can be inverted. Then

$$g = s_1 \oplus s_2 \ldots \oplus s_n$$

$\uparrow$ direct sum $\quad [s_n, s_l] = 0$

$$\begin{pmatrix} k & 0 & 0 \\ 0 & l & 0 \\ 0 & 0 & l \end{pmatrix}$$

If the Killing metric is negative definite the Lie algebra is compact.

The g_{ij} are use to lower indices

$$f_{ki}^{\ \ l} g_{lj} = f_{kij}$$

f_{kij} is completely antisymmetric

By definition $[X_i, X_j] = i f_{ij}^{\ \ k} X_k$

$\Rightarrow$ anti-symmetry in ij

we also have that

$$\langle ad(X_k) X_i, X_j \rangle + \langle X_i, ad(X_k) X_j \rangle = 0$$

$$\langle [X_k, X_i], X_j \rangle + \langle X_i, [X_k, X_j] \rangle = 0$$

$$f_{ki}^{\ \ l} g_{lj} + f_{kj}^{\ \ l} g_{li} = 0$$

$$f_{kij} + f_{kji} = 0$$

Then $f_{ijk} = -f_{jik} = f_{jki} = -f_{kji}$

g^{ij} is the inverse of the Killing metric

Casimir operator

$$C_2 = g^{ij}\underbrace{\hat{X}_i\hat{X}_j}_{\text{matrix rep of algebra}}$$

$$[C_2, \hat{X}_i] = 0$$

proof $[g^{ij}\hat{X}_i\hat{X}_j, \hat{X}_k]$

$$g^{ij}\hat{X}_i[\hat{X}_j, \hat{X}_k] + g^{ij}[\hat{X}_i, \hat{X}_k]X_j$$

$$g^{ij}\hat{X}_i\, i f_{jk}^{\ \ l}\hat{X}_l + g^{ij}\, i f_{ik}^{\ \ l}\hat{X}_l\hat{X}_j$$

$$= i g^{ij} f_{jk}^{\ \ l}\hat{X}_i\hat{X}_l + i \underset{ji}{\underset{\downarrow}{g^{ij}}} f_{\underset{\downarrow j}{i}k}^{\ \ l}\hat{X}_l\hat{X}_{\underset{\downarrow i}{j}}$$

$$i g^{ij} f_{jk}^{\ \ l}(\hat{X}_i\hat{X}_l + \hat{X}_l\hat{X}_i)$$

$$i f_{jkm}\, g^{ij} g^{ml}(\hat{X}_i\hat{X}_l + \hat{X}_l\hat{X}_i)$$

$$= i f_{jkm}(g^{ji}\hat{X}_i g^{ml}\hat{X}_l + g^{ml}\hat{X}_l g^{ji}\hat{X}_i)$$

$$= 0$$

Representations of Lie Algebras

We can get representations of the group from representations of the algebra

$$[X_i, X_j] = i f_{ij}^{k} X_k$$

a representation is a solution of $n \times n$ matrices $\hat{X}_i^J$ such that

$$[\hat{X}_i^J, \hat{X}_j^J] = i f_{ij}^{k} \hat{X}_k^J$$

Then $D^J(g(\xi)) = e^{i \xi^J \hat{X}_i^J}$

is a representation of the Lie group

$SU(2)$

We have to solve the relation

$$[J_1, J_2] = i J_3$$
$$[J_3, J_1] = i J_2 \qquad [J_2, J_3] = i J_1$$

raising and lowering operator

$$J_+ = J_1 + i J_2 \qquad J_- = J_1 - i J_2$$

then $[J_3, J_\pm] = [J_3, J_1 \pm i J_2]$

$$= -i J_2 \pm i(-i) J_1 = \pm(J_1 \pm i J_2)$$
$$= \pm J_\pm$$

if $J_3|jm\rangle = m\,|jm\rangle$

then $J_3 J_\pm |jm\rangle \sim (m\pm 1)\,|jm\rangle$

Representation is finite dimension

$\Rightarrow \exists$ highest weight state

which we call $|jj\rangle$ such that

$J_+ |jj\rangle = 0$

and $J_3 |jj\rangle = j\,|jj\rangle$

The we act with J_- on this state

$J_- |jj\rangle = c\,|j\,j-1\rangle$

again because the representation is finite dimensional, there should be a lowest weight state such that $\quad J_- |j\;j-p\rangle = 0$

we choose normalized states

$\langle jm|jm\rangle = 1$

$J_- |jm\rangle = c\,|j\,m-1\rangle$

$\langle jm|\,J_-^\dagger J_-\, jm\rangle = |c|^2$

$\langle jm|\,J^+ J_- |jm$

$$J_+ J_- = (J_1 + iJ_2)(J_1 - iJ_2)$$
$$= J_1^2 + J_2^2 + (-i)\, i J_3 - i.i J_3$$
$$= J_1^2 + J_2^2 + J_3^2 + J_3 - J_3^2$$

$J_1^2 + J_2^2 + J_3^2$ is casimir

$$[J_1, J_1^2 + J_2^2 + J_3^2] =$$
$$[J_1, J_2^2] + [J_1, J_3^2]$$
$$= J_2 \cancel{iJ_3} + i\cancel{J_3}J_2 + -J_3 \cancel{i}J_2$$
$$- i\cancel{J_2}J_3$$
$$= 0$$

Let us ~~act~~ act with J^2 on the highest weight stat

$$J^2 |j\, j\rangle = (J_- J_+ + J_3 + J_3^2)\, |j\, j\rangle$$
$$= (j + j^2)\, |j\, j\rangle$$

$$J_- J_+ = (J_1 - iJ_2)(J_1 + iJ_2)$$
$$= J_1^2 + J_2^2 + J_3^2 - J_3^2 - J_3$$

So j in $|j\, m\rangle$ is the j in $j(j+1)$ of the eigen-value of J^2

We still do not know if it is an integer or half integer

Then we act with J^2 on the lowest weight state

$$J^2 |j, j-p\rangle = (J_+ J_- - J_3 + J_3^2)|j, (j-p)\rangle$$

$$= \left(-(j-p) + (j-p)^2\right)|j, j-p\rangle$$

but this should have the same eigenvalue so

$$j(j+1) = -(j-p) + (j-p)^2$$

$$\Rightarrow \quad j(j+1) = -j + p + j^2 - 2pj + p^2$$

$$0 = -2j + p + p(p-2j)$$

$\Rightarrow$ either $p = 0$ trivial

or $p = 2j$

but p is an integer

$$\Rightarrow \quad j = \frac{p}{2}$$

SU(3)

Generators are traceless Hermition
3×3 matrices

diagonal m.e. of Hermitian matrices
are real, so we have two diagonal
generators. they are the equivalent
of σ_3

$$\lambda_3 = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix} \qquad \lambda_8 \doteq \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix}$$

SU(2) is a subgroup of SU(3). These
generators gives λ_1 and λ_2

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then there are 4 more generators

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

We can make again raising and
lowering operators (they have only one
nonzero me equal to 1

$$t_\pm = \frac{1}{2}(\lambda_1 \pm \lambda_2)$$
$$v_\pm = \frac{1}{2}(\lambda_4 \pm \lambda_5)$$
$$u_\pm = \frac{1}{2}(\lambda_6 \pm \lambda_7)$$

To find the representations we work with objects that have the same commutations relations

$\lambda_k \rightarrow \Lambda_k$ $\quad T_\pm = \frac{1}{2}(\Lambda_1 \pm i\Lambda_2)$

$V_\pm = \frac{1}{2}(\Lambda_4 \pm i\Lambda_5)$

$U_\pm = \frac{1}{2}(\Lambda_6 \pm i\Lambda_7)$

commutation relations with the diagonal generators

$[\Lambda_3, T_\pm] = \pm 2 T_\pm$ $\quad [\Lambda_8, T_\pm] = 0$

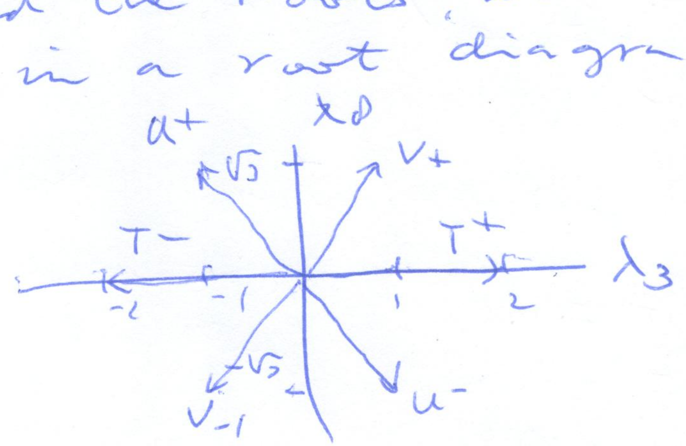
$[\Lambda_3, V_\pm] = \pm V_\pm$ $\quad [\Lambda_8, V_\pm] = \pm\sqrt{3}\, V_\pm$

$[\Lambda_3, U_\pm] = \mp U_\pm$ $\quad [\Lambda_8, U_\pm] = \pm\sqrt{3}\, U_\pm$

So we have again the ladder operators as for $SU(2)$ and we can use those to construct representations

The eigenvalues are called the weights

The amounts by which the ladder operators changes the weights are called the roots, we can draw them in a root diagram



Now the highest weight state has two quantum numbers $(\lambda_3^h, \lambda_8^h)$

they cannot be increased further by

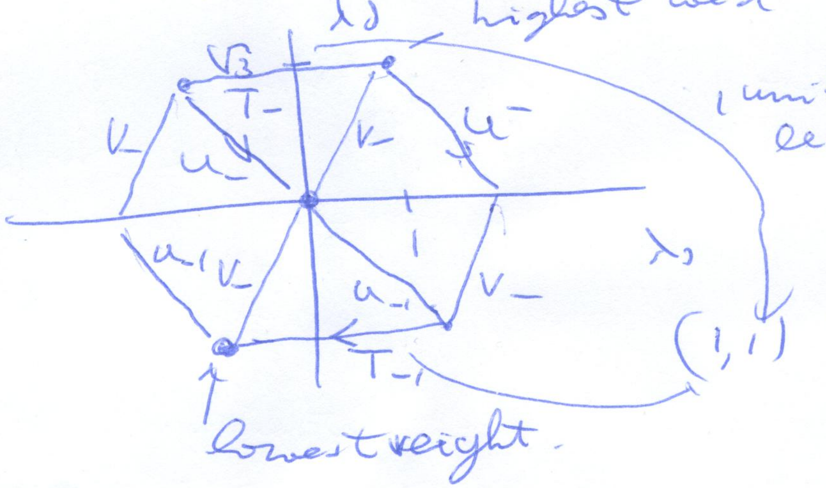
U_+, T_+ and V_+ So $U_+(\lambda_3^h, \lambda_8^h) = 0$

$T_+(\lambda_3^h, \lambda_8^h) = 0$

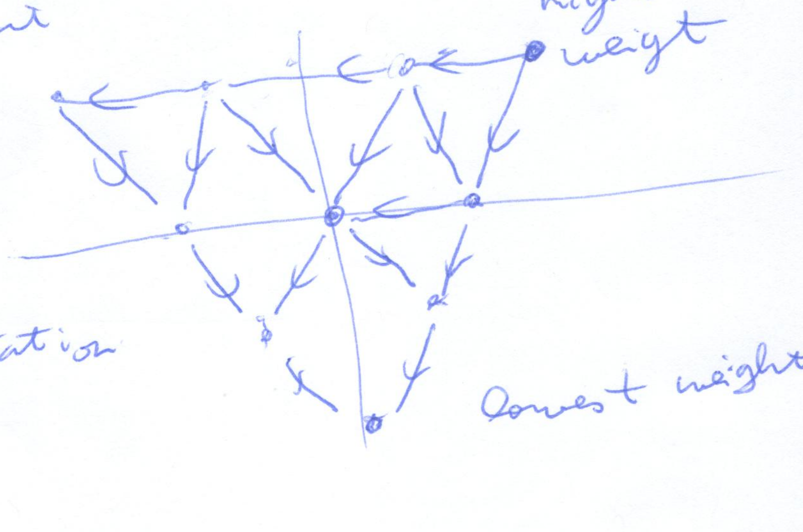
$V_+(\lambda_3^h, \lambda_8^h) = 0$

all other states are obtained by applying lowering operators to the highest weight state.

8 dimensional representation



10 d representat



General Lie algebra

This procedure can be generalized to any simple Lie algebra.

- find the maximum algebra of commuting generators. This is the Cartan subalgebra. They are denoted by h_i

- Next construct the ladder operators

$$[h_i, e_\alpha] = \alpha_i e_\alpha \quad \alpha_i > 0 \text{ raising} \qquad [h_i, \underbrace{e_\alpha^+}_{= e_{-\alpha}}] = -\alpha_i e_\alpha^+ \quad \text{lowering}$$

convention such that the α_i are real

Some properties

$$\alpha_i \langle e_\alpha, h_j \rangle = \langle ad(h_i) e_\alpha, h_j \rangle$$
$$= -\langle e_\alpha, ad(h_i) h_j \rangle$$
$$= \langle e_\alpha, [h_i, h_j] \rangle = 0$$

$$\Rightarrow \langle e_\alpha, h_j \rangle = 0$$

$$(\alpha_i + \beta_i) \langle e_\alpha, e_\beta \rangle = \langle ad(h_i) e_\alpha, e_\beta \rangle + \langle e_\alpha, ad(h_i) e_\beta \rangle$$
$$= 0$$

$$\Rightarrow \langle e_\alpha, e_\beta \rangle = 0 \quad \text{unless } \alpha_i + \beta_i = 0$$

$\Rightarrow \langle e_\alpha, e_\alpha \rangle = 0$

we need β such that $\langle e_\alpha, e_\beta \rangle \neq 0$
otherwise the Killing form would
be nondegenerate. This can only
be $e_{-\alpha}$

$\Rightarrow e_{-\alpha}$ must be a root
This is the lowering operator if e_α
is the raising operator

- We define $g_{ij} = \langle h_i, h_j \rangle$

g_{ij} is nondegenerate for semisimple
Lie algebra g^{ij} is inverse

$$\alpha \cdot \beta \equiv g^{ij} \alpha_i \beta_j$$

- Jacobi identity

$$[h_i, [e_\alpha, e_\beta]] = [h_i, e_\alpha] e_\beta + e_\alpha [h_i, e_\beta] - (\alpha \leftrightarrow \beta)$$

$$= \alpha_i e_\alpha e_\beta + \beta_i e_\alpha e_\beta - (\alpha \leftrightarrow \beta)$$

$$= (\alpha_i + \beta_i) \, [e_\alpha, e_\beta]$$

$[e_\alpha, e_\beta] \neq 0 \Rightarrow \alpha_i + \beta_i$ is also a root

– $[e_\alpha, e_{-\alpha}]$ commutes with all h_i

$\Rightarrow$ $[e_\alpha, e_{-\alpha}]$ = linear combination of h_i because cartan algebra was maximal.

$$\langle h_i [e_\alpha, e_{-\alpha}]\rangle = \langle h_i, ad(e_\alpha) e_{-\alpha}\rangle$$

$$= -\langle ad_\alpha(h_i), e_{-\alpha}\rangle = -\langle [e_\alpha, h_i], e_{-\alpha}\rangle$$

$$= \alpha_i \underbrace{\langle e_\alpha, e_{-\alpha}\rangle}_{\neq 0}$$

$$\Rightarrow [e_\alpha, e_{-\alpha}] \neq 0$$

$$\Rightarrow [e_\alpha, e_{-\alpha}] = \sum_i c_i h_i$$

$$\langle h_i, [e_\alpha, e_{-\alpha}]\rangle = \langle \sum_j h_i, \sum c_j h_j\rangle$$

$$= \sum_j c_j g_{ij} = \alpha_i \langle e_\alpha, e_{-\alpha}\rangle$$

$$= c_j = g^{-1}_{jk} \alpha_k \langle e_\alpha, e_{-\alpha}\rangle$$

$$= g^{jk} \alpha_k \langle e_\alpha, e_{-\alpha}\rangle$$

$$= \alpha^k \underbrace{\langle e_\alpha, e_{-\alpha}\rangle}$$

normalized as $\frac{2}{(\alpha \cdot \alpha)}$

$$\Rightarrow [e_\alpha, e_{-\alpha}] = \frac{2\alpha^i h_i}{(\alpha \cdot \alpha)}$$

$$h_\alpha \equiv \sum_i \frac{2\alpha^i}{\alpha \cdot \alpha} h_i$$

$$[h_\alpha, e_{\pm\alpha}] = \pm 2\, e_{\pm\alpha}$$

– $$[h_\alpha, e_\beta] = \frac{2(\alpha \cdot \beta)}{\alpha \cdot \alpha} e_\beta$$

$\Rightarrow$ h_α can have only integer eigenvalues (same as for $SU(2)$)

$\Rightarrow$ $\frac{2(\alpha \cdot \beta)}{\alpha \cdot \alpha}$ must be integer.

Based on this we can classify all semisimple Lie algebras

– For each root α there is only only e_α. Suppose there are two also e_α'. Then we can choose

$$\langle e_\alpha', e_{-\alpha} \rangle = 0 \quad \Rightarrow \quad [e_{-\alpha}, e_\alpha'] = 0$$

$\langle h_i, [e_{-\alpha}, e_\alpha'] \rangle = \alpha_i \langle e_\alpha', e_{-\alpha} \rangle$ $\Rightarrow$ e_α' is lowest weight

$\Rightarrow [e_{-\alpha}, e_\alpha'] = -\alpha^i h_i \langle e_\alpha', e_{-\alpha} \rangle$ not possible because

$= 0$ $\qquad [h_\alpha, e_\alpha'] = 2 e_\alpha'$