Unitary symplectic group

$$Sp(n) = Sp(2n, \mathbb{C}) \cap U(2n)$$

Kramers degeneracy

$\sigma_k \qquad \sigma_k^* = -C^{-1} \sigma_k \cdot C$

$C = i\sigma_2$

↑ charge conjugation matrix

spin orbit coupling $H = \vec{L} \cdot \vec{S}$

$\vec{S} = \frac{\hbar}{2} \sigma_k \qquad \vec{L} = \vec{r} \times \vec{p}$

$\vec{L}^* = -L$

$$H^* = L^* \cdot S^* = + L C^{-1} S C = C^{-1} H C$$

$\Rightarrow [C K, H] = 0$

↑ complex conjugation

$$(CK)^2 = i\sigma_2 K i\sigma_2 K = i^2 \sigma_2^2 K^2 = -1$$

$$\Rightarrow (\psi, CK\psi) = (CK\psi, (CK)^2\psi)^*$$

$$= -(CK\psi, \psi)^*$$

$$= -(\psi, CK\psi)$$

CK is anti-unitary

$\Rightarrow (\psi, CK\psi) = 0$

if $H\psi = E\psi$ then $HCK\psi = CK H\psi = CK E\psi$

$\Rightarrow$ $CK\psi$ is eigenstate

it is independent of $\psi \Rightarrow$

states are doubly degenerate.

This is Kramers degeneracy

SU(2)

$$u = \begin{pmatrix} x_0 + i x_3 & i x_1 + x_2 \\ i x_1 - x_2 & x_0 - i x_3 \end{pmatrix}$$

$$|x_0 + i x_3|^2 + |i x_1 + x_2|^2 = 1$$

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

$$\Rightarrow \quad SU(2) \cong \text{3 sphere}$$

SU(2) rows are perpendicular

$$(x_0 + i x_3)^* (i x_1 - x_2) + (i x_1 + x_2)^* (x_0 - i x_3)$$

$$= + i \cancel{x_0 x_1} - \cancel{x_0 x_2} + \cancel{x_3 x_1} + i \cancel{x_2 x_3}$$

$$- i \cancel{x_1 x_0} + \cancel{x_2 x_0} - \cancel{x_1 x_3} - i \cancel{x_2 x_3} = 0$$

in terms of Pauli σ-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$u = x_0 \mathbb{1} + i x_k \sigma_k$$

$$[\sigma_i, \sigma_j] = 2 i \varepsilon_{ijk} \sigma_k \qquad \sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij}$$

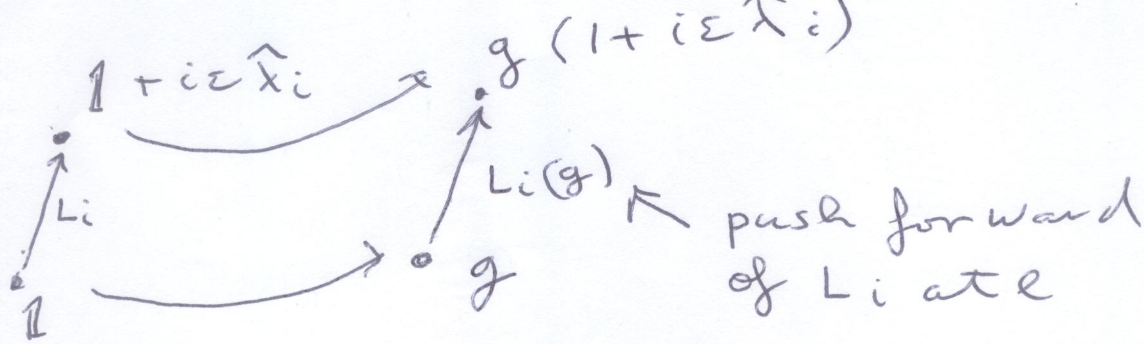
the $i\sigma_k$ are the generators of SU(2)

$$[i\sigma_k, i\sigma_l] = - f_{kl}^{\ \ m} \, i \sigma_m$$

$$f_{kl}^{\ \ m} = \varepsilon_{klm}$$

structure constants of the group

Invariant vector fields



$L_i(g)$ is in the tangent space of TG_g

$L_i \in TG_e$

Example $SU(2)$

$$g(\mathbb{1} + i\varepsilon\sigma_3) = x_0 + i\sigma_1 x_1 + i\sigma_2 x_2 + i\sigma_3 x_3$$

$$+ x_0 i\varepsilon\sigma_3 + - x_1 \varepsilon \sigma_3\sigma_1 - x_2\varepsilon\sigma_2\sigma_3 - x_3\varepsilon\sigma_3\sigma_3$$

$$= (x_0 - \varepsilon x_3) + i\sigma_1(x_1 - \varepsilon x_2) + i\sigma_2(x_2 + \varepsilon x_1) + i\sigma_3(x_3 + \varepsilon x_0)$$

$$L_3 F(g) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(F(g(1 + i\varepsilon\sigma_3)) - F(g)\right)$$

$$= -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} + x_0 \frac{\partial}{\partial x_3}$$

$$L_3 = \varepsilon_{3kl}\, x_k \frac{\partial}{\partial x_l} + x_0 \frac{\partial}{\partial x_3}$$

in general

$$L_p = \underbrace{\varepsilon_{pkl}\, x_k \frac{\partial}{\partial x_l}}_{\text{angular momentum operator}} + x_0 \frac{\partial}{\partial x_p}$$

$$x_1 \to x_1 - \varepsilon x_2$$
$$x_2 \to x_2 + \varepsilon x_1$$
$$x_3 \to x_3 + \varepsilon x_0$$

Then $x_0 = \sqrt{1 - x_1^2 - x_2^2 - x_3^2}$

$$\to \sqrt{\underbrace{1 - x_1^2 - x_2^2 - x_3^2}_{x_0^2}} + \cancel{2\varepsilon x_1 x_2} - \cancel{2\varepsilon x_1 x_2} - 2\varepsilon x_0 x_3$$

$$= x_0 \left(1 - \frac{2\varepsilon x_0 x_3}{x_0^2}\right)^{\frac{1}{2}} = x_0 - \varepsilon x_3$$

which is exactly what it should be

$$[L_p, L_q]$$

$$\left[L_p, x_0 \frac{\partial}{\partial x_q}\right] = \varepsilon_{pke} x_k \frac{\partial}{\partial x_e} x_0 \frac{\partial}{\partial x_q}$$

$$- x_0 \frac{\partial}{\partial x_q} \varepsilon_{pke} x_k \frac{\partial}{\partial x_e}$$

$$\partial_{x_e} x_0 = -\frac{x_e}{x_0}$$

$$= -\overbrace{\varepsilon_{pke} \frac{x_k x_e}{x_0}}^{=0} \frac{\partial}{\partial x_q}$$

$$- x_0 \, \varepsilon_{pql} \frac{\partial}{\partial x_e}$$

$$\left[x_0 \frac{\partial}{\partial x_p}, L_q\right] = -x_0 \, \varepsilon_{pqe} \frac{\partial}{\partial x_e}$$

$$\mathcal{L}_{p} = \varepsilon_{pke} x_k \frac{\partial}{\partial x_e}$$

$$\{[\mathcal{L}_p, \mathcal{L}_q] = -2 \, \varepsilon_{pqe} \mathcal{L}_e$$

$$\Rightarrow [L_p, L_q] = -2 \, \varepsilon_{pqe} L_e$$

This work for all Lie groups in general $[L_i, L_j] = -f_{ij}^{\ \ k} L_k$

Time reversal invariance for spin $\frac{1}{2}$

time reversal $\vec{S} \to -\vec{S}$

in quantum mechanics $T S T^{-1} = -S$

$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$

$$T \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = e^{i\delta} \left|\tfrac{1}{2} -\tfrac{1}{2}\right\rangle$$

$\sigma_+ = (\sigma_1 + i\sigma_2)/2 \qquad \sigma_+ \left|\tfrac{1}{2} -\tfrac{1}{2}\right\rangle = \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$

$\sigma_- = (\sigma_1 - i\sigma_2)/2 \qquad \sigma_- \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = \left|\tfrac{1}{2} -\tfrac{1}{2}\right\rangle$

$T\sigma_- T^{-1} = \sigma_+ \qquad T\sigma_+ T^{-1} = -\sigma_-$

$$T \sigma_- \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = T \left|\tfrac{1}{2} -\tfrac{1}{2}\right\rangle$$

$$\parallel$$

$$-\sigma^+ T \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle = -\sigma^+ e^{i\delta} \left|\tfrac{1}{2} -\tfrac{1}{2}\right\rangle$$

$$= -e^{i\delta} \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

$$\Rightarrow T \left|\tfrac{1}{2} -\tfrac{1}{2}\right\rangle = -e^{i\delta} \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle$$

$$\Rightarrow T = e^{i\delta} i \sigma_2 K \, (-1)$$

$$T\sigma_+ T^{-1} = T(\sigma_1 + i\sigma_2) T^{-1} = -\sigma_1 + i\sigma_2$$
$$= -\sigma_-$$

$$T\sigma_- T^{-1} = T(\sigma_1 - i\sigma_2)^{-1} = -\sigma_1 - i\sigma_2$$
$$= -\sigma_+$$

Note that $T i T^{-1} = -i$

The exponential map

Vector field $X^\mu \partial_\mu$

corresponding flow

$$\frac{dx^\mu}{dt} = X^\mu(x(t))$$

Let L_i be the vector field corresponding to $i\lambda_i$ and

$$g(t) = e^{it\lambda_i} = 1 + it\lambda_i - \frac{1}{2}t^2\lambda_i^2 + \dots$$

$$g(t+\varepsilon) = e^{i((t+\varepsilon)\lambda_i)} = e^{it\lambda_i} e^{i\varepsilon\lambda_i}$$

$$= g(t)(1 + i\varepsilon\lambda_i)$$

$$\Rightarrow \frac{dg}{dt} = \frac{1}{\varepsilon}\left(g(t+\varepsilon) - g(t)\right)$$

$$= \frac{1}{\varepsilon}\left(g(t)(1+i\varepsilon\lambda_i) - g(t)\right)$$

$$\equiv L_i\, g(t)$$

$$\Rightarrow g(t) = e^{L_i t}$$

with initial condition $g(0) = 1$

Right invariant vector fields

multiplication on the right to push forward a group element

$$(1 + i\varepsilon\sigma_3)\, g = g + i\varepsilon R_3\, g$$

$$[R_i, R_j] = 2\,\varepsilon_{ijk}\, R_k$$

in general $[R_i, R_j] = f_{ij}{}^{k} R_k$

$$[L_i, L_j] = -f_{ij}{}^{k} R_k$$

$$[L_i, R_j] = 0$$

Maurer Cartan form

$$g = x^0 + i x^k \sigma^k$$
$$g^{-1} = x^0 - i x^k \sigma^k$$

we calculate $dg\, g^{-1}$

$$dg = dx^0 + i dx^k \sigma^k$$

$$dx^0 = -\frac{x_1 dx_1}{x_0} - \frac{x_2 dx_2}{x_0} - \frac{x_3 dx_3}{x_0}$$

$$dg\, g^{-1} = i\sigma_1 \left(\left(x_0 + \frac{x_1^2}{x_0}\right) dx_1 + \left(x_3 + \frac{x_1 x_2}{x_0}\right) dx_2 + \left(x_2 + \frac{x_1 x_3}{x_0}\right) dx_3 \right.$$
$$+ i\sigma_2 (\cdots) + i\sigma_3 (\cdots)$$

$$R_1 = x^0 \partial_1 + x^3 \partial_2 - x^2 \partial_3$$

$$dg\, g^{-1} = \omega_R = i\sigma_i\, \omega_R^i$$

Maurer - Cartan forms

$$\omega_R^1 = \left(x_0 + \frac{x_1^2}{x_0}\right) dx_1 + \left(x_3 + \frac{x_1 x_2}{x_0}\right) dx_2$$
$$\left(- x_2 + \frac{x_1 x_3}{x_0}\right) dx_3$$

evaluate $\omega_R^1 (R^1)$

$dx_1(\partial_1) = 1$
$dx_1(\partial_2) = 0$
$dx_1(\partial_3) = 0$

$$\Rightarrow \omega_R^1 (R_1) =$$
$$\left(x_0 + \frac{x_1^2}{x_0}\right) x_0 + \left(x_3 + \frac{x_1 x_2}{x_0}\right) x_3 + \left(-x_2 + \frac{x_1 x_3}{x_0}\right)(-x_2)$$
$$= x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$$

see beginning of chapter 11

$$\omega_R^1(R_2) = \left(\cancel{x_0} + \frac{x_1^2}{\cancel{x_0}}\right)(-x_3) + \left(\cancel{x_3} + \cancel{\frac{x_1 x_2}{x_0}}\right)x_0 + \left(-x_2 + \cancel{\frac{x_1 x_3}{x_0}}\right)x_1$$

$$= 0$$

$$-x^3\partial_1 + x^0\partial_2 + x^1\partial_3$$

$$\Rightarrow \omega_R^k(R_l) = \delta_{kl}$$

Left invariant Maurer–Cartan form

$$g^{-1}dg = \omega_L = i\sigma_k\,\omega_L^k$$

$$\omega_L^i(L_j) = \delta_{ij}$$

$$d\omega_R = d(dg\,g^{-1}) = \overset{\to 0}{d^2 g} \wedge g^{-1} - dg \wedge dg^{-1}$$

$$g^{-1}g = 1 \Rightarrow dg^{-1}\,g + g^{-1}dg = 0$$

$$\Rightarrow dg^{-1} = -g^{-1}dg\,g^{-1}$$

$$d\omega_R = dg \wedge g^{-1}dg\,g^{-1} = dg\,g^{-1} \wedge dg\,g^{-1} = \omega_R \wedge \omega_R$$

$d^2 = 0$

$$\omega_R \wedge \omega_R = \omega^i_R \wedge \omega^j_R\, i\sigma_i\, i\sigma_j = \frac{1}{2}\omega^i_R \wedge \omega^j_R\,[i\sigma_i, i\sigma_j] = -\frac{1}{2}\omega^i_R \wedge \omega^j_R\, f_{ij}^{\ \ k}\, i\sigma_k$$

Maurer Cartan relations

$$d\omega^k_R = -\frac{1}{2} f_{ij}^{\ \ k}\,\omega_R^i \wedge \omega_R^j$$

For the left invariant forms we obtain

$$d\omega_L^k = \frac{1}{2} f_{ij}^{\ \ k}\,\omega_L^i \times \omega_L^j$$