Lie groups

These are continuous groups of matrices. They can be compact or noncompact. The group looks the same at any point of the group manifold, and usually the group is analyzed near the identity.

$$g = \mathbb{1} + \varepsilon G$$

$$g^n = (1 + \varepsilon G)^n = \left(1 + \frac{n\varepsilon}{n} G\right)^n \underset{n\to\infty}{=} e^{\varepsilon n G}$$

so we can get a group element anywhere by multiplying them

Simplest Lie group: $U(1)$

$$\{ e^{i\varphi} \} \qquad e^{i\varphi_1} e^{i\varphi_2} = e^{i(\varphi_1 + \varphi_2)}$$

Examples of Lie groups

$GL(n, \mathbb{C})$ invertible real $n \times n$ matrices
$GL(n, \mathbb{R})$ " complex $n \times n$ "

$SL(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ with $\det A = 1$
$SL(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ with $\det A = 1$

$A = 1 + \varepsilon G$ then $\det A = 1 + \varepsilon \, \mathrm{Tr}\, G$
$\Rightarrow$ G is traceless $\Rightarrow$ 1 parameter less

$GL(n, \mathbb{C})$ is $2n^2$ dimensional
$SL(n, \mathbb{C})$ is $2n^2 - 1$ dimensional

Unitary group

$n \times n$ matrices with complex matrix elements such that $U^+U = 1$

$\Rightarrow \det U^+U = 1 \Rightarrow$

$\det U^+ \det U = 1$

$\overset{\shortparallel}{(\det U)^*} \Rightarrow |\det U| = 1$

$\Rightarrow \det U = e^{i\varphi}$

infinitesimally

$$U = 1 + \varepsilon A$$

$$U U^+ = 1 = 1 + \varepsilon(A + A^+) + O(\varepsilon^2)$$

$$\Rightarrow A = -A^+$$

diagonal m.e. are purely imaginary
off-diagonal m.e are complex

so group has $n + 2\frac{n(n-1)}{2}$ parameters

$$U^+U = 1 \Rightarrow \sum_j U^+_{kj} U_{jk} = 1$$

$$\sum U^*_{jk} U_{jk} = 1$$

$$\Rightarrow |U_{jk}| < 1 \quad \Rightarrow U(n) \text{ is compact}$$

$U(n)$ is <u>not simple</u> because $U(1)$ is an invariant subgroup

$SU(n) \subset U(n)$ with $\det U = 1$

then $U(1)$ is only subgroup if $e^{i\varphi n} = 1$
that is why it is sometimes called simple because it has not continuous subgroups

The Orthogonal group

$n \times n$ real matrices with $O^T O = 1$

$\det O^T \det O = 1 \quad \Rightarrow \det^2 O = 1$

$\Rightarrow \det O = \pm 1$

$\det O = 1 \;:\; SO(n)$

infinitesimally

$O = 1 + \varepsilon A \quad \Rightarrow \quad O^T O = 1 + \varepsilon (A^T + A) + O\varepsilon^2$

$\Rightarrow A^T = -A$

$\Rightarrow$ diagonals are zero

$A \in \mathbb{R} \quad \Rightarrow \quad \frac{n(n-1)}{2}$ parameters

Symplectic group

$\omega^T = -\omega \qquad \omega$ fixed

$Sp(2n, \mathbb{R}) = \{ S \in Gl(2n, \mathbb{R}) \mid S^T \omega S = \omega \}$

$Sp(2n, \mathbb{C}) = \{ S \in Gl(2n, \mathbb{C}) \mid S^T \omega S = \omega \}$

ω can be brought to standard form

$$\omega = \begin{pmatrix} 0 & \alpha_1 & & \\ -\alpha_1 & 0 & & \\ & & 0 & \alpha_2 \\ & & -\alpha_2 & 0 \end{pmatrix}$$

then $S^T \begin{pmatrix} \begin{pmatrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & \alpha_2 \\ -\alpha_2 & 0 \end{pmatrix} \end{pmatrix} S = \begin{pmatrix} \begin{pmatrix} 0 & \alpha_1 \\ -\alpha_1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & \alpha_2 \\ -\alpha_2 & 0 \end{pmatrix} \end{pmatrix}$

$$= S^T \begin{pmatrix} \sqrt{\alpha_1} & & & \\ & \sqrt{\alpha_1} & & \\ & & \sqrt{\alpha_2} & \\ & & & \sqrt{\alpha_1} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}}_{\equiv \omega_0} \begin{pmatrix} \sqrt{\alpha_1} & & & \\ & \sqrt{\alpha_1} & & \\ & & \sqrt{\alpha_1} & \\ & & & \sqrt{2} \end{pmatrix} S$$

$$S' = \begin{pmatrix} \sqrt{\alpha_1} & & & \\ & \sqrt{\alpha_1} & & \\ & & \sqrt{\alpha_2} & \\ & & & \sqrt{2} \end{pmatrix} S \Rightarrow S'^T = S^T \begin{pmatrix} \sqrt{\alpha_1} & & & \\ & \sqrt{\alpha_1} & & \\ & & \sqrt{\alpha_2} & \\ & & & \sqrt{\alpha_2} \end{pmatrix}$$

$$\Rightarrow \quad S'^T \omega_0 S' = \begin{pmatrix} \sqrt{\alpha_1} & & & \\ & \sqrt{\alpha_1} & & \\ & & \sqrt{\alpha_2} & \\ & & & \sqrt{\alpha_2} \end{pmatrix} \omega_0 \begin{pmatrix} \sqrt{\alpha_1} & & & \\ & \sqrt{\alpha_1} & & \\ & & \sqrt{\alpha_2} & \\ & & & \sqrt{\alpha_2} \end{pmatrix}$$

$$S'' = S' \begin{pmatrix} \sqrt{1/\alpha_1} & & & \\ & \sqrt{1/\alpha_1} & & \\ & & \sqrt{1/\alpha_2} & \\ & & & \sqrt{1/\alpha_2} \end{pmatrix}$$

$$\Rightarrow \quad S''^T \omega_0 S'' = \omega_0$$

by permutation ω_0 can also be chosen as

$$\omega_0 \to \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

$$S = 1 + \varepsilon A \Rightarrow$$

$$(1 + \varepsilon A^+)\, \omega_0 \,(1 + \varepsilon A) = \omega_0$$

$$\Rightarrow \quad \varepsilon (A^T \omega_0 + \omega_0 A) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

$$\begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix}$$

$$\begin{pmatrix} c^T & -a^T \\ d^T & -b^T \end{pmatrix} + \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = 0$$

$\Rightarrow \quad c^T = c \qquad b^T = \cancel{-}b$

$\quad a^T = -d \qquad d^T = -a$

$\Rightarrow$ generator is of the form

$\begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$ with $\begin{matrix} b^T = b \\ c^T = c \end{matrix}$

$Sp(2n, \mathbb{R})$ has $n^2 + 2\frac{1}{2}n(n+1)$ parameters

$= 2n^2 + n$ parameter

$\Rightarrow$ generator is traceless

$\det(S^T \omega S) = \det \omega = 1$

$\det S^T \det \omega \det S = 1 \Rightarrow \det S = \pm 1$

$\overset{\shortparallel}{\det S}$

$\det S = -1$ is not a group

$\Rightarrow \det S = +1$ for $Sp(2n, \mathbb{R})$