Group algebra

$$x = x_1 g_1 + x_2 g_2 + \cdots x_{|G|} g_{|G|}$$

$$x_\mu \in \mathbb{C}$$

$$x = x_1 g_1 + x_2 g_2$$
$$y = y_1 g_1 + y_3 g_3$$

$$xy = x_1 y_1 g_1^2 + x_1 y_3 g_1 g_3 + x_2 y_1 g_2 g_1 + x_2 y_3 g_2 g_3$$

The group algebra is a representation.

$$D(g)\, g_i = g\, g_i = g_j D_{ji}(g)$$

regular representation

matrix with one nonzero element (it is 1) in each column and row

only the identity has 1 on the diagonal $\Rightarrow$ $\chi^R(g) = |G|$ if $g = e$

$= 0$ if $g \neq e$

The regular representation is reducible

$$\chi^R_i = \sum_\nu a_\nu \chi^\nu_i$$ conjugacy class

apply to identity

$$\Rightarrow |G| = \sum_\nu a_\nu \dim_\nu$$

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using the orthogonality of characters

$$a_\nu = \frac{1}{|G|} \sum d_i \chi^R_i \chi^{\nu *}_i$$

only nonzero for identity class

$$= \frac{1}{|G|} \sum_{i = \text{identity}} 1 \, |G| \, \dim_\nu = \dim_\nu$$

$$\Rightarrow |G| = \sum (\dim_\nu)^2$$

Symmetry groups in Quantum Mechanics

group acts on degrees of freedom
x which also can include spin

$x \to gx$

Then wave function transforms
linearly $\psi \to D(g)\psi$.

This follows from the linearity of
the Schrödinger equation - otherwise
the quantum mechanics of the transformed
wave function would be different

$x \to g_1 g_2 x$ then $\psi \to D(g_1 g_2)\psi(x)$

$x \to g_1(g_2 x)$ the $\psi \to D(g_1)\psi(g_2 x)$

$\to D(g_1) D(g_2) \psi(x)$

$\Rightarrow D(g_1 g_2) = D(g_1) D(g_2)$

$\Rightarrow$ D is a reprensentation of the group.

example rotations $x \to Ox$

then $\psi \to D^J_{mm'} \psi_{m'}$

Wigner D-matrices

when G is a symmetry, D has
to be unitary or anti-unitary
because all scalar products should
have the same absolute value.

Then $D(g) H = H D(g)$

Then $H D(g) \psi = D(g) H \psi$

if $D(g)$ is irreducible representation then H does not move the states out of this representation. So the Hilbert space can be decomposed as

$$\mathcal{H} = \mathcal{H}_{\gamma_1} \oplus \cdots \oplus \mathcal{H}_{\gamma_n}$$

Because $D(g)$ is irreducible on $\mathcal{H}_\gamma$ is an invariant subspace according to Schur's lemma, we must have that

$$H = \lambda \mathbb{1} \text{ on } \mathcal{H}_\gamma$$

Of course $\lambda = E_\gamma$, the energy eigenvalue

<u>example</u> $O(3)$ symmetry.

$$[O, H] = 0 \qquad O \in O(3)$$

then all states with the same spin have the same symmetry.