Characters

$$\mathrm{Tr}\,(c^{-1} D\, c) = \mathrm{Tr}\, D$$

$\Rightarrow$ trace is the same for equivalent representations

$$\mathrm{Tr}(D(g_1^{-1} g\, g_1^{-1})) = \mathrm{Tr}(D(g))$$

$\Rightarrow$ Trace is the same for all elements in a conjugacy class

character of a representation

$$\chi(g) = \mathrm{Tr}\, D(g)$$

it is a class function

For an irreducible representation we have

$$\frac{1}{|G|}\sum_{g\in G} D^{\gamma *}_{ji}(g)\, D^{\kappa}_{ke}(g) = \frac{1}{\dim \gamma}\,\delta_{ik}\,\delta_{je}\,\delta^{\gamma\kappa}$$

make $i = \cancel{k}\, j$ and sum, $k\, \cancel{j} = e$

$$\frac{1}{|G|}\sum_{g\in G} \mathrm{Tr}\, D^{\gamma}(g)\, \mathrm{Tr} D^{\kappa}(g) = \frac{1}{\dim \gamma}\,\dim \gamma\, \delta^{\gamma\kappa}$$

$$= \frac{1}{|G|}\sum_{i} d_i \left(\chi_i^{\gamma}\right)^{*} \chi_i^{\kappa}$$

$\uparrow$ # element in i^{th} conjugacy class characters of irreducible representations are orthogonal and complete on the space of class functions.

This implies that there are as many different characters of irreps as there are conjugacy classes.

e is always a conjugacy class
e is mapped to the identity matrix
$\Rightarrow \quad \chi^J(e) = \dim J$

completeness relation

check constant by summing over K after $K = \ell$

$$\frac{1}{|G|} \sum_J \chi^{J*}_K \, \chi^J_\ell = \cdots \frac{1}{d_K} \delta_{K\ell}$$

take $g, h = e \Rightarrow \sum_J (\chi^J)^2 = |G|$

Let us study the irreps of S_3

$|S_3| = 3! = 6$

conjugacy classes $\quad (1) \quad (12)\ (23)\ (13) \quad (123)\ (132)$

So we have three independent characters of irreps

We are going to make a character table

	(1)	(12) (23) (13)	(123) (132)	
χ_1	1	1	1	
χ_2	2	0	-1	
χ_3	3	-1	0	

χ_1 is the character of the identity representation

$$\sum \chi_k^2(e) = 6 = 1 + d_2 + d_3$$

only possibility is $d_2 = 2 \quad d_3 = 3$

we also know $\sum d_K \chi_K^* \chi_K = |S_3| = 6$

for χ_2 we have

$$4 + |\chi_{(12)}|^2 3 + |\chi_{123}|^2 2 = 6$$

$$\Rightarrow \chi_{(12)} = 0 \qquad \chi_{123} = \pm 1$$

orthogonality $\Rightarrow \chi_{(123)} = -1$

For χ_3 the only possibility $= 1$

$$\chi_{3(12)} = -1 \qquad \chi_3(123) = 0$$

Completeness of characters on the space of class functions

ρ is representation of G on V

$$\rho : V \to V$$

Let φ be a class function on G

$$\varphi : G \to \mathbb{C} \qquad \varphi(hgh^{-1}) = \varphi(g)$$

$$\rho(\varphi) \equiv \sum_{g \in G} \varphi(g)\, \rho(g)$$

then $\rho(\varphi)$ commutes with G for every representation V

$$\rho(h)\rho(\varphi)\rho(h^{-1}) = \sum_{g \in G} \varphi(g)\, \rho(hgh^{-1})$$

$$= \sum_{g \in G} \underbrace{\varphi(kgk^{-1})}_{\varphi(g)}\, \rho(g)$$

$$= \rho(\varphi)$$

$$\Rightarrow \quad \rho(h)\rho(\varphi) = \rho(\varphi)\rho(h) \qquad \forall h \in V$$

if $\rho(h)$ is irreducible then by Schur's lemma $\rho(\varphi) = \lambda \mathbb{1}$

assume now that φ is orthogonal to all irreducible characters

$$\langle \varphi, \chi_k \rangle = 0$$

$$\operatorname{Tr} \rho(\varphi) = \sum_{g \in G} \varphi(g)\, \chi_V(g) = 0$$

$$\rho(\varphi) = \lambda \mathbb{1} \to \lambda = 0 \Rightarrow \varphi = 0$$

Let us now try S_4

$$\sum \chi_\mu^2(e) = |S_4| = 24$$

conjugacy classes $(1), (12), (123), (12)(34), (1234)$

$$\Rightarrow \sum_{\mu=1}^{5} \chi_\mu^2(e) = 24$$

identity rep is 1d

$$\Rightarrow \quad 1 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 24$$

4 dimensional rep not possible because 7 cannot be written as the sum of 3 squares

So $a1 + b4 + c9 = 23$ and $a + b + c = 4$

Solution

a	b	c
1	4	0
0	2	1

$c = 0$ not possible because we need too many 1d an 2d irreps $\Rightarrow c = 1, 2$

$$c = 1 \Rightarrow \left.\begin{array}{l} a + 4b = 14 \\ a + b = 3 \end{array}\right\} \text{ no solution}$$

$$c = 2 \quad \left.\begin{array}{l} a + 4b = 5 \\ a + b = 2 \end{array}\right\} \Rightarrow \begin{array}{l} a = 1 \\ b = 1 \end{array}$$

	(1)	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	± 1	± 2	0
χ_4	3	± 1	0	± 1	± 1
χ_5	3	± 1	0	± 1	± 1
$d(\ )$	1	6 $\binom{4}{2}$	$2\cdot 4 = 8$	$3 = \frac{6}{2}$	$6 = 3\cdot 2 = \binom{4}{2}$

$$\sum d_x |\chi_4|^2 = 24$$

$$9 + 6\chi_4^2((12)) + \chi_4^2((123))\cdot 8 + 3\chi^2((12)(34)) + 6\chi^2((1234)) = 24$$

solution $\chi_4^2((12)) = 1 \quad \chi_4^2((123)) = 0$

$\chi_4^2((12)(34)) = 1 \quad \chi^2((1234)) = 1$

The sign of the permutation is also a 1d representation

then we also know $\chi_3^2(\cdot)$

orthogonality $\Rightarrow$ $\chi_3((123)) = -1$, $\chi_3((12)(34)) = 2$