Lecture 2

Last ~~time~~ : - Functional derivatives
use definition

- Euler Lagrange

- First integral

- Lagrange multipliers

Tody : - Catenary

- field theory

- action principle

- Noether's theorem

Action principle

action $S = \int_{t_1}^{t_2} L \, dt$

L Lagrangian

classical equations of motion are the stationary point of S:

$\delta S = 0$ $\qquad L = L(q_i, \dot{q}_i, t)$

$\Rightarrow$ Euler-Lagrange: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Example: Central force problem

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

EL eqs $\quad r: \quad m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$

$\theta \quad \underbrace{\frac{d}{dt}(mr^2\dot{\theta}) = 0}_{\text{angular momentum conservation}}$

First integral $\quad E = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$\frac{dE}{dt} = 0$$

Noether's theorem

For each conservation law there is a symmetry.
For the central potential we had energy conservation and angular momentum conservation. Noether's theorem makes the relation between conservation law and symmetry explicit.

$$S = \int_0^T \left(\tfrac{1}{2} m (\dot r^2 + r^2 \dot\theta^2) - V(r) \right) dt$$

integrand is invariant for

$$\theta(t) \to \theta(t) + \varepsilon \alpha$$

$\leftarrow$ constant

Since S is stationary under variations of θ, we have that S is stationary under

$$\theta(t) \to \theta(t) + \varepsilon(t)\,\alpha$$

$\leftarrow$ now depends on time.

$$\delta S = \int_0^T dt\, \tfrac{1}{2} m r^2 (\dot\theta + \dot\varepsilon \alpha)^2 - \int_0^T \tfrac{1}{2} m r^2 \dot\theta^2$$

$$= \int_0^T dt\, \tfrac{1}{2} m r^2 \alpha\, 2 \dot\varepsilon \dot\theta$$

assume that $\varepsilon(0) = \varepsilon(T) = 0$

$$= - \int_0^T dt\, m \frac{d}{dt}(r^2 \dot\theta)\, \varepsilon \quad \Rightarrow$$

$$\Rightarrow \quad \frac{d}{dt} r^2 \dot\theta = \text{constant}$$

(angular momentum conservation)

Many degrees of freedom

The action principle is also valid for field theories

field $\varphi(x_\mu)$

action $S(\varphi) = \int L\,dt = \int \underbrace{\mathcal{L}(x^\mu, \varphi, \partial_\mu\varphi)}_{\text{Lagrangian density}} d^{d+1}x$

$$\delta S = \int \left(\delta\varphi(x) \frac{\partial \mathcal{L}}{\partial \varphi(x)} + \delta \partial_\mu \varphi \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) d^{d+1}x$$

$$= \int \left(\delta\varphi \frac{\partial \varphi}{\partial \varphi} - \delta\varphi \frac{d}{dx_\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) d^{d+1}(x)$$

assume that $\delta\varphi$ vanishes on the boundaries

Euler Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x_\mu} \frac{\partial \mathcal{L}}{\partial \varphi_\mu}$$

Example: vibrating string

$$L = \int_0^L dx \left(\frac{1}{2}\rho \dot{y}^2 - \frac{1}{2} T y'^2 \right)$$

$'' \frac{dy}{dx}$

$$S = \int L\,dt$$

$$\text{EL:} \quad -\frac{d}{dt}\rho\dot{y} + \frac{1}{2}T\,2\frac{d}{dx}\frac{dy}{\partial x} = 0$$

$$\Rightarrow \rho\ddot{y} - T y'' = 0$$

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