Group action on sets

$G, \quad X$

$g \in G \qquad g : X \to X$

$$g_1(g_2 x) = (g_1 g_2) x$$
$$ex = x$$

- orbit of x: $\quad \{gx, g \in G\}$

- action of a group is <u>transitive</u>
  if $\{gx, g \in G\} = G$

- action of group is <u>faithful</u>
  if $\quad \forall x \in X \quad g(x) = x$ implies
  $g = e$

  So if $\quad g_1(x) = g_2(x)$
  then $\qquad$ then $g_2^{-1} g_1 x = x$
  $\Rightarrow \quad g_1 = g_2$

we could have a set $G_s < G$ that acts as identity on X. then G/G_s is faithful.

action of G is free if $\exists x$ $gx = x$ implies $g = e$, then g has no fixed points

Example: a group action is free an transitive, then $\{gx_0, g \in G\} = X$.

$$g_1 x_0 = g_2 x_0 \Rightarrow g_1^{-1} g_2 x_0 = x_0$$

$$\Rightarrow g_1^{-1} g_2 = e \Rightarrow g_1 = g_2$$

So in essence $G \cong X$

Example: a group action is transitive but not free

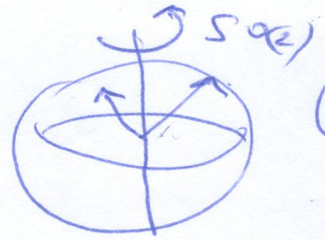
$$H = \{ g \in G \mid g x_0 = x_0 \}$$

$H \leq G$

$$\left.\begin{array}{l} g_1 x_0 = x_0 \\ g_2 x_0 = x_0 \end{array}\right\} \Rightarrow g_1 g_2 x_0 = x_0$$

$$g_1 x_0 = g_2 x_0 \Rightarrow g_1^{-1} g_2 \in H$$

So $X \sim G/H$



Example

$SO(3) : S_2 \to S_2$

SO(2) subgroup leaves north pole and South pole invariant

So $SO(3)/SO(2) \simeq S_2$

Representations of groups

homomorphism $D : G \to Gl(n, \mathbb{C})$

↑ $n \times n$ matrices with complex matrix elements

$$D(g_1)\, D(g_2) = D(g_1 g_2)$$
$$D(g^{-1}) = (D(g))^{-1}$$

equivalent representations

$$D'(g) = C^{-1} D(g) C$$

is also a representation $D' \sim D$

Real representations

$$D'(g) = (D(g))^*$$

– if $D^* \sim D$ and $\exists$ basis such that D becomes real the we have a real representation

- if $D^* \sim D$ and we cannont find a basis for which D is real then the representation is pseudo real.

<u>Example</u>: $SU(2)$ is pseudoreal

$U \in SU(2)$ $\qquad U = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$

$|a|^2 + |b|^2 = 1$

$U^+ U = 1$

we have that $U^* \sim U$

$$U^* = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

~~so~~ It is not possible to find a basis where U becomes real $\forall U \in SU(2)$ if $U \in \mathbb{R}$ then $a, b \in \mathbb{R}$ and $|a|^2 + |b|^2 = 1$ and it is not possible to describe a 3 parameter group by a two-parameter group.

Direct sum of representations

$$D^{(1)} \oplus D^{(2)} = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix}$$

Direct product of representations

$$[D^{(1)} \otimes D^{(2)}](g)(e^{(1)}_i \otimes e^{(2)}_j) = (e^{(1)}_k \otimes e^{(2)}_\ell) D^{(1)}_{ki}(g) D^{(2)}_{\ell j}(g)$$

$e^{(1)}_i$ basis for $D^{(1)}$

$e^{(2)}_i$ basis for $D^{(2)}$

Matrix elements of the tensor product

$$[D^{(1)}(g) \otimes D^{(2)}(g)]_{k\ell, ij} = D^{(1)}_{ki} D^{(2)}_{\ell j}$$

$$A \otimes C \cdot B \otimes D = AB \otimes CD$$

$$A_{ki} C_{\ell j} B_{im} D_{jn} = AB_{km} CD_{\ell n}$$

$$= AB \otimes CD_{km\ell n}$$

Irreducible representations

representations that cannot be further reduced into direct sums

irreducible subspace

A_α = set of maps on V

irreducible subspace $U \subseteq V$

is an invariant subspace $A_\alpha U = U$

and U can be only 1 or V.

nontrivial invariant subspace U

$$V = U \oplus U'$$

↑ complement

(not necessarily invariant)

if U' is invariant then the A_α are completely reducible

$$A_\alpha = \left(\begin{array}{c|c} X & 0 \\ \hline 0 & Y \end{array}\right)$$

Schur's Lemma

This is probably the most important lemma of representation theory.

A_α } set of linear operators on U
B_α } for A_α and V for B_α

both irreducible

$\Lambda : U \to V$ such that

intertwining operator

$$\Lambda A_\alpha = B_\alpha \Lambda$$

then i) $\Lambda = 0$

ii) Λ is one to one and onto

then $A_\alpha = \Lambda^{-1} B_\alpha \Lambda$

proof $a \in \operatorname{Ker} \Lambda$, then $A_\alpha a \in \operatorname{Ker} \Lambda$

$\Rightarrow \operatorname{Ker} \Lambda$ is invariant subspace of A_α

$b \in \operatorname{Im} \Lambda$ then $B_\alpha \Lambda c = \Lambda A_\alpha c$

$\Rightarrow b = \Lambda c$

$\Rightarrow B_\alpha b \in \operatorname{Im} \Lambda$

$\operatorname{Im} \Lambda$ is invariant subspace of B_α

but A_α and B_α are irreducible

$\Rightarrow$ either $\Lambda = 0$ or $\operatorname{Ker} \Lambda = 0$

$\operatorname{Im} \Lambda = V$

then Λ is invertible and onto.

Corollary

$\{A_\alpha\}$ acts irreducible on U

and $\Lambda : U \to U$ $\qquad U$ n dimensional

$$A_\alpha \Lambda = \Lambda A_\alpha$$

then $\Lambda = 0$ or $\Lambda = \lambda \mathbb{1}$

proof obviously $\Lambda = 0$ is possible

so when $\Lambda \neq 0$ consider $\Lambda - x\mathbb{1}$

we also have $A_\alpha(\Lambda - x\mathbb{1}) = (\Lambda - x\mathbb{1})A_\alpha$

$\det(\Lambda - x\mathbb{1})$ is polynomial of order n

fundamental theorem of algebra tells that it should have at lease one root

Say $x = \lambda$.

then $\Lambda - \lambda\mathbb{1}$ is not invertible

$\Rightarrow \Lambda - \lambda\mathbb{1} = 0$ by Schur's lemma

Unitary representations

G finite $\qquad D(g)$ representation

$D(g) : V \to V$

(x, y) inner product on V

new inner product $\quad \langle x, y \rangle = \frac{1}{|G|} \sum_{g \in G} (D(g)x, D(g)y)$

we have that $\quad \langle D(g)x, D(g)y \rangle = \langle x, y \rangle$

$\forall (x, y)$

$\Rightarrow \quad D^{+}(g) D(g) = \mathbb{1}$ for orthonormal basis

$\Rightarrow D(g)$ is a unitary representation

does not work for non compact groups because the group average may not be convergent.

$\Rightarrow$ representations of finite groups are unitary

Orthogonality relation

$D^{J} : V_J \to V_J \qquad D^{K} : V_K \to V_K$

$M : V_K \to V_J \qquad D^{J}, D^{K}$ irreps

$$\Lambda = \sum_{g \in G} D^{J}(g^{-1}) \, M \, D^{K}(g)$$

$$\Lambda \overset{K}{D}(g_0) = \sum_{g \in G} D^{J}(g^{-1}) M D^{K}(g) D^{K}(g_0)$$

$$= \sum_{g \in G} D^{J}(g^{-1}) M D^{K}(g g_0)$$

$$= \sum_{g \in G} D^{\gamma}(g_0 g^{-1}) M D^{\kappa}(g g_0)$$

$$= D^{\gamma}(g_0) \Lambda$$

$\gamma = \kappa$ then Schur's lemma

$$\sum_{g \in G} D^{\gamma}_{ij}(g^{-1}) M_{jk} D^{\gamma}_{kl} = \lambda(M) \delta_{il}$$

$\gamma \neq \kappa$, D^{γ} and D^{κ} inequivalent
then this gives zero according to Schur's lemma

$$\Rightarrow \sum_{g \in G} D^{\gamma}_{ij}(g^{-1}) M_{jk} D^{\gamma}_{kl}(g) = \lambda(M) \delta_{il} \delta^{\gamma\kappa}$$

choose M as Matrix with only $M_{jk} \neq 0$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{\gamma\kappa} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \sum_{g \in G} D^{\gamma}_{ij}(g^{-1}) D^{\gamma}_{kl}(g) = \lambda_{jk} \delta_{il} \delta^{\gamma\kappa}$$

put $i = l$ and sum over i

$$\Rightarrow |G| \delta_{jk} = \lambda_{jk} \dim \gamma$$

$$\Rightarrow \frac{1}{|G|} \sum_{g \in G} D^{\gamma}_{ij}(g^{-1}) D^{\kappa}_{kl}(g) = \frac{1}{\dim \gamma} \delta_{jk} \delta_{il} \delta^{\gamma\kappa}$$

$$\|$$

$$(D^{\gamma}_{ji}(g))^*$$