Parsevel's theorem

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x)$$

u_n orthonormal set

$$\|f\|^2 = \int f^* f \, dx =$$

$$= \sum_{nm} \int a_n a_m \underbrace{u_n^* u_m \, dx}_{\to \delta_{nn}}$$

$$= \sum_n \|a_n\|^2$$

Example $\quad u_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ on $[-\pi, \pi]$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{igx}$$

$$\|f\|^2 = \int_{-\pi}^{\pi} \frac{1}{2\pi} dx = 1$$

$$\frac{1}{\sqrt{2\pi}} e^{igx} = \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{\sqrt{2\pi}}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{igx - inx} = \frac{\sin \pi(g-n)}{\pi(g-n)}$$

Parseval : $\displaystyle \sum_n \frac{\sin^2 \pi(g-n)}{(\pi(g-n))^2} = 1$

$$\Rightarrow \quad \sum_n \frac{\sin^2 \pi g}{(\pi(g-n))^2} = 1$$

$$\text{or} \quad \sum_n \frac{1}{\pi^2 (g-n)^2} = \frac{1}{\sin^2 \pi g}$$

Orthogonal Polynomials

inner product $(u, v) = \int_a^b w(x)\, u(x)\, v(x)\, dx$

$$P_0(x) = \frac{1}{\|1\|_w}$$

$$p_n(x) = \sum_{k=0}^{n} a_k x^k$$

Orthogonal polynomial $(P_n, P_m) = \delta_{nm}$

Construction:

$$P_{n+1} = \alpha\, x P_n + \sum_{k=0}^{n} a_k P_k$$

assume that $P_0, \dots, P_n$ are already an orthogonal set

$(P_{n+1}, P_k) = 0$ for $k < n$

$$\Rightarrow \quad \alpha (P_k, x P_n) + a_k = 0$$

$$\Rightarrow \quad P_{n+1} = \alpha\, x P_n + \sum_{k=0}^{n} (-\alpha)(P_k, x P_n) P_k$$

α is fixed by the normalization of P_{n+1}

$$(P_{n+1}, P_{n+1}) = \alpha^2 (x P_n, x P_n) - 2\alpha^2 \sum_{k=0}^{n} (P_k, x P_n)^2 + \sum_{k=0}^{n} \alpha^2 (P_k, x P_n)^2 = 1$$

Three step recursion relation

Orthogonal polynomials satisfy a 3 step recursion

$$x\,p_n(x) = b_n\,p_{n+1}(x) + a_n\,p_n(x) + b_{n-1}\,p_{n-1}(x)$$

Proof: $(x p_n, p_k)$ $\quad k \le n-2$

$= (p_n, \underbrace{x\,p_k}_{\text{at most order } n-1})$

$\Rightarrow \quad = 0 \quad \Rightarrow$ q.e.d.

Weierstraß approximation theorem

$f(x)$ on $[a,b]$ continuous

then $\exists$ polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon \quad \forall x \in [a,b]$$

$\Rightarrow$ orthogonal polynomials are complete

Examples of orthogonal polynomials

Legendre polynomials: Orthogonal polynomials on $[-1,1]$ with $w = 1$

The Legendre Polynomials satisfy the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof: P_n is polynomial of order n

$$\int_{-1}^{1} (P_n P_k)\, dx \qquad k < n$$

$$= \int_{-1}^{1} \frac{1}{2^n n!} \frac{1}{2^k k!} \frac{d^n}{dx^n}(x^2-1)^n \frac{d^k}{dx^k}(x^2-1)^k$$

partial integrate until ∂_x^k factor vanishes. Boundary terms vanish

$k = n$ partial integrate n times

then we get $\displaystyle\int_{-1}^{1} \frac{1}{(2^n n!)^2} (x^2-1)^n (-1)^n \underbrace{\frac{d^{2n}}{dx^{2n}} x^{2n}}_{(2n)!}$

$$= \frac{(2n)!}{2^{2n} n! n!} \int_{-1}^{1} (-1)^n (x^2-1)^n dx = \frac{2}{2n+1}$$

Recursion

$$(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$$

Hermite polynomials

; $\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x) \sim \delta_{nm}$

Generating function $e^{2tx-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$

$$\| $$

$$e^{-(t-x)^2+x^2}$$

$$\| $$

$$\sum_n e^{x^2} \frac{1}{n!} \frac{d^n}{dt^n} e^{-(t-x)^2} = \sum_n \frac{e^{x^2}(-1)^n}{n!} \frac{d^n}{dx^n} e^{-(t-x)^2}$$

$$\frac{H_n}{n!} = \frac{1}{n!}\frac{d^n}{dt^n} e^{2tx-t^2}\Big|_{t=0} = \frac{e^{x^2}(-1)^n}{n!} e^{-x^2}$$

$$H_n = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Orthogonality relation

$$\int H_n H_m e^{-x^2} dx = 2^n n! \sqrt{\pi}\, \delta_{nm}$$

__proof__ choose $m > n$

$$\int (-1)^{n+m} \cancel{e^{-x^2}} \cancel{e^{x^2}} \frac{d^n}{dx^n} e^{-x^2} e^{x^2} \underbrace{\frac{d}{dx^m} e^{-x^2}}_{m\text{'th order polynomial}}$$

vanishes after partial integrating n times.

Tshebychef polynomials

Second Kind $\int_{-1}^{1} \sqrt{1-x^2}\, T_n(x) T_m(x)\, dx \propto \delta_{nm}$

First Kind $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}\, U_n(x) U_m(x) \propto \delta_{nm}$

They follow from $\cos m\theta$ and $\sin n\theta$ orthogonal sets

$$\int_0^{\pi} \cos n\theta \cos m\theta \, d\theta = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \underbrace{\cos n \cos^{-1} x}_{T_n(x)} \cos m \cos^{-1} x$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$$\int_0^{\pi} \sin n\theta \, \cancel{\cos} \sin m\theta \, d\theta = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \quad \underbrace{\sin n \cos^{-1} x}_{\text{not a polynomial}} \sin m \cos^{-1} x$$

$$\sin \cos^{-1} x = y \Rightarrow \sin^2(\cos^{-1} x) + \underbrace{\cos^2(\cos^{-1} x)}_{x^2} = 1$$

$$\Rightarrow \sin \cos^{-1} x = \sqrt{1-x^2}$$

$$\doteq \int_{-1}^{1} dx \sqrt{1-x^2} \underbrace{\frac{\sin n \cos^{-1} x}{\sqrt{1-x^2}}}_{U_n(x)} \frac{\sin m \cos^{-1} x}{\sqrt{1-x^2}}$$

Distributions and testfunctions

Linear operator in functionspace

$$f(x) = \int_a^b A(x,y)\, f(y)\, dy$$

like $f_i = A_{ij} f_j$

δ - function

$$f(x) = \int \delta(x-y)\, f(y)\, dy$$

derivative of δ - function δ'

we consider δ' as a linear operator

$$\int \delta'(x-y)\, f(y)\, dy = \int -\frac{d}{dy}\left(\delta(x-y)\right) f(y)$$

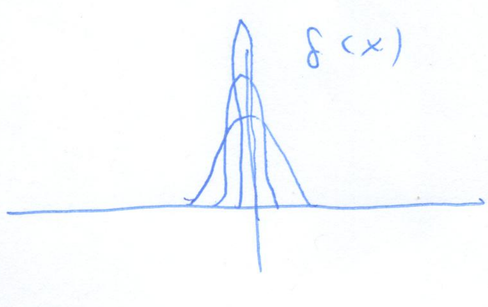
$$= + \int \delta(x-y)\, f'(y)\, dy = + f'(x)$$

$$\Rightarrow \int \delta'(-y)\, f(y)\, dy = f'(0)$$

$$\int \delta'(y)\, \underbrace{f(-y)}_{\substack{\text{arbitrary} \\ = g(y)}}\, dy = \underset{g'(0)}{f'(0)}$$

in general $\int \delta^n(x)\, f(x) = (-1)^n f^{(n)}(0)$

δ - function as limi



$\delta(x)$

$\delta'(x)$

Distributions are function with discontinuities or derivatives thereof. There are defined by making them act on a set of nice functions, called test function. Eg the functions that are infinitely differentable and vanish sufficiently fast at infinity

S. the distributions are the dual space to the linear space of test function

$$(\delta, \varphi) = \varphi(0)$$

$$(\delta', \varphi) = -\varphi'(0)$$

$$\delta(ax-b) = \frac{1}{|a|}\delta\left(x-\frac{b}{a}\right)$$

we can also use them to define a <u>weak</u> derivative

$$\int v(x)\,\varphi(x)\,dx = -\int u(x)\,\varphi'(x)\,dx$$

then $v(x)$ is the weak derivative of $u(x)$

e.g. $\int \frac{d}{dx}\,\text{sign}\,x\;\varphi(x) = -\int \text{sign}\,x\;\varphi'(x)\,dx$

Example weak derivative of $\log|x|$

$$I = -\int_{-\infty}^{\infty} \varphi'(x) \log|x| = -\lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0}} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon'}^{\infty}\right) \varphi'(x) \log|x| \, dx$$

$$\varphi' \log|x| = \frac{d}{dx}(\varphi \log|x|) - \frac{1}{x}\varphi(x) \qquad x \neq 0$$

test functions vanish at $\pm\varepsilon$

$$I = \varphi(\varepsilon')\log|\varepsilon'| - \varphi(-\varepsilon)\log|\varepsilon| + \lim_{\substack{\varepsilon\to 0\\ \varepsilon'\to 0}} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon'}^{\infty}\right) \frac{\varphi(x)}{x} dx$$

special case $\varepsilon = \varepsilon' \to 0$

then $(\varphi(\varepsilon) - \varphi(-\varepsilon))\log|\varepsilon| \sim 2\varphi'(0)\,\varepsilon\log|\varepsilon| \to 0$

$$\Rightarrow (-\log|x|, \varphi') = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x}dx$$

$$\parallel$$

$$\left(\frac{d}{dx}\log|x|, \varphi\right)$$

$$\Rightarrow \frac{d}{dx}\log|x| = P\left(\frac{1}{x}\right)$$

Principal value integral

weak derivative of $\log|x|$