Function spaces

$C^n[a,b]$ nfold differentiable functions on $[a,b]$

$C^\infty[a,b]$ Analytic function on $[a,b]$

A function space is a linear vectorspace

$\lambda f(x) + \mu g(x)$ is also a function

To find the distance between two functions we need a norm. There are many different norms, eg

$\int_a^b |f(x)|\, dx$

A norm should satisfy:

i) positivity $\|f\| \geq 0$, $\|f\| = 0$ implies $f = 0$

ii) triangle inequality $\|f+g\| \leq \|f\| + \|g\|$

iii) linear homogeneity $\|\lambda f\| = |\lambda|\, \|f\|$

i) $\int_a^b |f(x)|\, dx$ satisfies i)

ii) $\int_a^b |f(x)+g(x)|\, dx \leq \int_a^b (|f(x)| + |g(x)|)\, dx = \|f\| + \|g\|$

iii) is obvious

Convergence

We often have to approximate a function $f(x)$ by a sequence $f_n(x)$ on D

There are different ways of converging.

i) Pointwise convergence: $\forall x \in D$, the sequence $f_n(x)$ converges to $f(x)$

ii) Uniform convergence

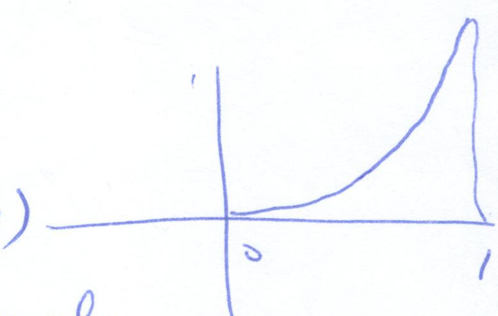
$$\sup_{x \in D} |f_n(x) - f(x)| \to 0 \quad \text{for } n \to \infty$$

iii) Convergence in the mean

$$\int_D |f_n(x) - f(x)| \, dx \to 0 \quad \text{for } n \to \infty$$

Uniform convergence is important because it is a condition for interchanging integrals and limits.

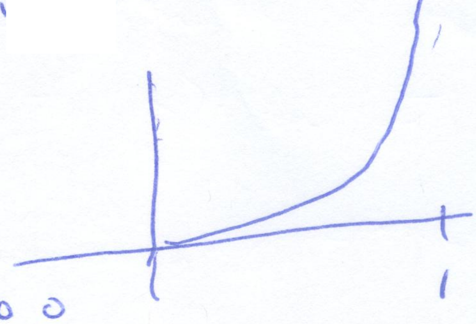
Example 1) $f_n = x^n$



- $f_n \to 0$ pointwise on $[0,1)$
- does not converge uniformely

on $[0,1]$ it does also not converge pointwise but converges in the mean

Example 2 $\quad f_n(x) = (n+1)\,x^n$



$$\int_0^1 f_n(x) = 1$$

- converges pointwise to 0 on $[0, 1)$
- does not converges uniformely on $[0, 1)$

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$$

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0$$

Almost all notion

With convergene in the Lebegue sense an function converges to a function in almost all pints, i.e. $\forall x \in D$ with the exception of a set of zero measure.

norm for Lebesgue Space

L^p norm

$$\|f\|_p = \left(\int |f(x)|^p dx \right)^{1/p}$$

Most common norm $\quad p = 2$

Complete space

Cauchy sequence

$\forall_{\varepsilon > 0} \quad \exists_N$ such that $\forall_{n, m > N} \quad \|f_n - f_m\| < \varepsilon$

a normed vector space is complete if each Cauchy sequence converges to some element of the space.

Banach space a normed complete vector space

$L^p[a,b]$ space are complete when the norm is interpreted as Lebesgue integral

Hilbert space Banach space $L^2[a,b]$

in general, a Hilbert space is a Banach space where the norm is derived from the inner product.

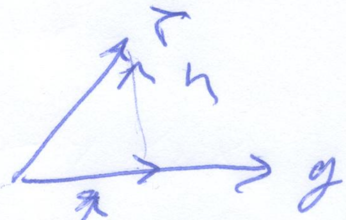
For $L^2[a,b]$

$$\|f\| = \left(\int_a^b f(x)\, f(x)\, dx\right)^{1/2}$$

$$= (f, f)^{1/2}$$

Proof of Cauchy-Schwartz inequality

$$|(f,g)| \le \|f\|\,\|g\|$$



define $h = f - \dfrac{(f,g)\,g}{(g,g)}$

then $(h,g) = (f,g) - (f,g) = 0$

$\Rightarrow h \perp g$

aply Pythagorus to h and $\dfrac{(f,g)}{(g,g)}\,g$

$$\|f\|^2 = \|h\|^2 + \frac{|(f,g)|^2}{\|g\|^4}\|g\|^2$$

$$\Rightarrow |(f,g)|^2 \geq \|f\|^2\|g\|^2 \Rightarrow \text{qed.}$$

Triangle inequality

$$\|f+g\| \le \|f\| + \|g\|$$

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + (f,g) + (g,f)$$

$$= \le \|f\|^2 + \|g\|^2 + 2|(f,g)|$$

$$= (\|f\| + \|g\|)^2$$

Consequence of Cauchy Schwartz inequality

$f_n \to f$ ie $\|f_n - f\| \to 0$

then $(f_n, g) \to (f,g)$

Orthonormal function sets

$$(u_n, u_m) = \delta_{nm}$$

Example a) $u_n = e^{2\pi i n x}$

then $\int_0^1 dx\, (u_n, u_m) = \delta_{nm}$

b) $u_m = \sqrt{2} \sin m\pi x$

u_n is a complete set on $(0,1)$

Any function f in the Hilbert space can be expanded as

$$f = \sum_{n=0}^{\infty} a_n u_n(x)$$

$$\Rightarrow (u_m, f) = \sum_n a_m (u_m^*; u_n) = a_n$$

Best approximation

$$\Delta = \| f - \sum_{n=1}^{N} a_n u_n \|^2$$

we want to minimize Δ

$$0 = \|f\|^2 - \sum a_n (f, u_n) - \sum a_n^* (u_n, f) + \underbrace{\sum_{nm} a_m^* a_n (u_m, u_n)}_{|a_n|^2}$$

$$\Rightarrow 0 = \|f\|^2 - \sum_{n=1}^{N} |(f, u_n)|^2 + \sum_{n=1}^{N} |a_n - (u_n, f)|^2$$

$\Rightarrow$ minimized if $a_n = (u_n, f)$