1. Functional Derivatives

Functional $\mathcal{F} : C^{\infty}(\mathbb{R}) \to \mathbb{R}$ ← real numbers

$C^{\infty}(\mathbb{R})$: smooth functions $f : \mathbb{R} \to \mathbb{R}$

derivative $f(x + \delta x) = f(x) + \delta x\, f'(x) + \cdots$

Taylor expand.

For functional derivative we also use the definition of the derivative

$$\mathcal{F}(\underbrace{f + \delta f}_{\text{also a function}}) = \mathcal{F}(f) + \underbrace{\frac{\delta \mathcal{F}}{\delta f}}_{\substack{\text{functional}\\ \text{derivative}}} \cdot \delta f + \cdots$$

Example $\mathcal{F}(f) = \int_{-\infty}^{\infty} f^2(x)\, dx$

$$\mathcal{F}(\underbrace{f + \delta f}_{\text{function}}) = \int_{-\infty}^{\infty} \left(f(x) + \delta f(x)\right)^2 dx$$

$$= \underbrace{\int_{-\infty}^{\infty} f^2(x)}_{\mathcal{F}(f)} + \int_{-\infty}^{\infty} 2 f(x)\, \delta f x$$

$$\Rightarrow \quad \frac{\delta \mathcal{F}}{\delta f(x)} = 2 f(x)$$

Let us now do a more complicated example

$$\mathcal{F}(f) = \int_{-\infty}^{\infty} [f'(x)]^2 \, dx$$

$$\mathcal{F}(f+\delta f) = \int_{-\infty}^{\infty} (f'(x) + \delta f'(x))^2 \, dx$$

$$= \int_{-\infty}^{\infty} (f'(x))^2 \, dx + \int_{-\infty}^{\infty} 2 f'(x) \, \delta f'(x) \, dx$$

$$= \mathcal{F}(f) + \underbrace{2 f'(x) \, \delta f \Big|_{-\infty}^{\infty}}_{\text{use } \delta f \text{ that vanish at } \pm\infty} + (-) \int_{-\infty}^{\infty} 2 f''(x) \, \delta f(x) \, dx$$

$$2 f'(x) \, \delta f \Big|_{-\infty}^{\infty} = 0$$

$$\Rightarrow \quad \frac{\delta \mathcal{F}(f)}{\delta f(x)} = -2 f''(x)$$

Let us now do the general case

$$\mathcal{F} = \int_{-\infty}^{\infty} L(f, f') \, dx$$

L depends on f and f'

$$\mathcal{F}(f+\delta f) = \int_{-\infty}^{\infty} L(f+\delta f, f'+\delta f') \, dx$$

$$= \int_{-\infty}^{\infty} L(f, f') \, dx + \int_{-\infty}^{\infty} \left(\frac{\partial L}{\partial f} \delta f + \frac{\partial L}{\partial f'} \delta f' \right) dx$$

$$= \mathcal{F}(F) + \int_{-\infty}^{\infty} \left(\frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'} \right) \delta f$$

$$+ \underbrace{\int_{-\infty}^{\infty} d\left(\frac{\partial L}{\partial f'} \, \delta f \right)}_{\substack{\text{vanishes because} \\ \delta f \to 0 \text{ for } x \to \pm\infty}}$$

$$\Rightarrow \quad \frac{\delta \mathcal{F}}{\delta F} = \frac{\partial L}{\partial f} - \frac{d}{dx} \frac{\partial L}{\partial f'}$$

The same derivation works if the endpoints are x_1 and x_2 and $\delta f(x_1) = \delta f(x_2) = 0$ (fixed endpoints)

Application : Euler-Lagrange equations

action $\quad S = \int_{t_1}^{t_2} L(x(t), x'(t)) \, dt$

The Euler-Lagrange-equations are the stationary points of the action

$$\frac{\delta S}{\delta x} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial x(t)} - \frac{d}{dt} \frac{\partial L}{x'(t)}$$

<u>example</u> $L = \frac{1}{2}\dot{x}^2 - V(x)$

$$\frac{\delta L}{\partial x} = - V'(x)$$

$$\frac{\delta L}{\delta \dot{x}} = \dot{x}$$

EL equation $\quad V'(x) + \ddot{x} = 0$

or: $\quad \ddot{x} = - V'(x)$

Newton equation of motion

<u>First integral of EL equations</u>

Assume that L does not have an explicit time dependence.

The $\quad \frac{\partial L}{\partial t} = 0 \quad L(x(t), x'(t), t) = L(x(t), x'(t))$

Then $\frac{d}{dt}\left(L - x' \frac{\partial L}{\partial x'}\right)$

$$= \frac{\partial L}{\partial x} x' + \cancel{\frac{\partial L}{\partial x'} x''} - \cancel{x'' \frac{\partial L}{\partial x'}} - (x')^2 \frac{\partial^2 L}{\partial x \partial x'} - x' x'' \frac{\partial^2 L}{\partial x'^2}$$

$$= x' \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial x'} \right)$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{=0 \text{ by EL equations}}$$

$L - x' \frac{\partial L}{\partial x'}$ is called the first integral.

What is it for $L = \frac{1}{2}\dot{x}^2 - V(x)$?

$$\frac{1}{2}\dot{x}^2 - V(x) - \dot{x} \underbrace{\frac{\partial L}{\partial \dot{x}}} = +\frac{1}{2}\dot{x}^2 - V(x) - \dot{x}^2$$

$$= -\frac{1}{2}\dot{x}^2 - V(x)$$

this is minus the energy

Lagrange multipliers

road $g(x,y) = 0$

contour map of hill
$h = f(x,y)$

what is the higlest point on the road?

$df = 0 \qquad df = \underbrace{\vec{\nabla} f}_{(\partial_x f,\, \partial_y f)} \cdot \vec{dr}$

and $\vec{dr}$ should be on the road

$dg = 0 \qquad dg = \vec{\nabla} g \cdot \vec{dr} = 0$

$\Rightarrow$ $\vec{\nabla} f$ and $\vec{\nabla} g$ are parallel

$$\nabla f - \lambda \nabla g = 0$$

$$g(x, y) = 0$$

λ is called a Lagrange multiplier

<u>n constraints</u> $\quad g_1 = \cdots = g_n = 0$

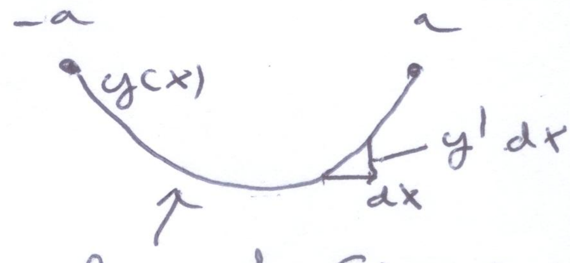
then $dg_k = 0 = \vec{\nabla} g_k \cdot d\vec{r}$

$$df = \nabla \vec{f} \cdot \vec{dr} = 0$$

$$\Rightarrow \quad \vec{\nabla} f = \sum_{k=1}^{\infty} \lambda_k \nabla g_k$$

Lagrange mulipliers.

The Catenary



shape is called the **catenary**

$$\text{Length} = \int_{-a}^{a} \sqrt{1+y'^2}\,dx$$

$$\text{energy} = \int_{-a}^{a} \rho\sqrt{1+y'^2}\,y\,dx$$

we have to find the stationary point of

$$\int_{-a}^{a} \rho\sqrt{1+y'^2}\,y\,dx - \lambda\int_{-a}^{a}\sqrt{1+y'^2}\,dx$$

This is a functional of y and y'.
So we can use the EL equations

$$\rho\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{\rho y}{\sqrt{1+y'^2}}\,y' - \frac{\lambda y'}{\sqrt{1+y'^2}}\right) = 0$$

First integral $\quad L - x' \frac{\partial L}{\partial x'} = \text{const}$

$$\rho\sqrt{1+y'^2}\,y - \lambda\sqrt{1+y'^2} - y'\left(\frac{\rho y y' - \lambda y'}{\sqrt{1+y'^2}}\right) = \text{const}$$

$\Rightarrow$

$$(\rho y-\lambda)(1+\cancel{y'^2}) - \cancel{y'^2(\rho y-\lambda)} = c\sqrt{1+y'^2}$$

$$\Rightarrow \quad \rho y - \lambda = c\sqrt{1+y'^2}$$

put $\quad \rho = 1$

$$y = \lambda + b_1\cosh(x\,b_2)$$

$$b_1\cosh x b_2 = c\sqrt{1+b_1^2 b_2^2\sinh^2 x b_2}$$

$$b_1 b_2 = 1 \qquad = c\cosh x$$

$$c = b_1$$

$$y = \lambda + c\cosh\frac{x}{c}$$

$$y(\pm a) = 0 \quad \Rightarrow \quad \lambda + c\cosh\frac{a}{c} = 0$$

$$\int_{-a}^{a}\sqrt{1+y'^2}\,dx = \ell = \int_{-a}^{a}\left(1+\sinh^2\left(\frac{x}{c}\right)\right)^{1/2}dx$$

$$= \int_{-a}^{a}\cosh\frac{x}{c}\,dx$$

$$2c\sinh\frac{a}{c} = \ell$$

$$\frac{\lambda^2}{c^2} - \frac{\ell^2}{4c^2} = 1$$

$$\lambda = \sqrt{c^2 + \frac{\ell^2}{4}}$$

$$y = \sqrt{c^2 + \frac{\ell^2}{4}} + c\cosh\frac{x}{c}$$

$$2c\sinh\frac{a}{c} = \ell$$

$$\frac{\lambda^2}{c^2} - \frac{\ell^2}{4c^2} = 1 \qquad \cancel{\Rightarrow \lambda = \frac{\ell}{2}}$$

$$\cancel{y = \frac{\ell}{2}} \qquad \lambda = \sqrt{c^2 + \frac{\ell^2}{4}}$$

$$y = \sqrt{c^2 + \frac{\ell^2}{4}} + c\cosh\frac{x}{c}$$

$$2c\sinh\frac{a}{c} = \ell$$