Solutions of Homework set #2

1) $H = \begin{pmatrix} 0 & c \\ c^+ & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & c \\ c^+ & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\Rightarrow \quad \underbrace{c\, b = 0}_{\substack{m \text{ eqs} \\ n \text{ unknowns}}} \quad \text{and} \quad \underbrace{c^+ a = 0}_{\substack{n \text{ eqs} \\ m \text{ unknowns}}}$

take $m > n$

this $m-n$ nontrivial solutions

has only $a = 0$ as solution

$\Rightarrow \exists\ m-n$ zero modes generically eigenvectors of the form $\begin{matrix} 0 \\ \vdots \\ 0 \\ b_1 \\ \vdots \\ b_n \end{matrix}$ $\}m$

2) $\vec{a} \times \vec{b} = \varepsilon_{ijk}\, a_j b_k$

rotation $\quad a_j' = \sigma_{jj'} a_{j'}$

$b_k' = \sigma_{kk'} b_{k'}$

$\Rightarrow \vec{a}' \times \vec{b}' = \varepsilon_{ijk}\, \sigma_{jj'} \sigma_{kk'}\, a_{j'} b_{k'}$

$\Rightarrow \sigma_{i'i} \cdot (\vec{a}' \times \vec{b}')_{i'} = \underbrace{\varepsilon_{ijk}\, \sigma_{i'i}\, \sigma_{jj'}\, \sigma_{kk'}}_{\det \sigma = 1 \text{ for rotation}} a_{j'} b_{k'}$

$\sigma^T = \sigma^{-1}$

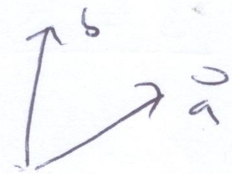
$= \varepsilon_{i j' k'}\, a_{j'} b_{k'}$

$\Rightarrow (\vec{a}' \times \vec{b}')_{i'} = \sigma_{i'i}\, \varepsilon_{ij'k'}\, a_{j'} b_{k'}$

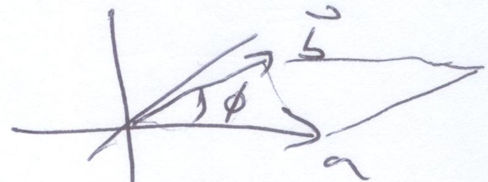
$(\vec{a}' \times \vec{b}')_{i'} = \sigma_{i'i} (a \times b)_i$

$\Rightarrow$ length of $|a \times b|$ is invariant under rotations

2b)



Make a rotation so that $\vec{a}$ and $\vec{b}$ are in the xy plane



$\vec{a}$ // x axis

area spanned by $\vec{a}$ and $\vec{b}$ is $|\vec{a}||\vec{b}|\sin\phi$

outer product of $\vec{a}$ and $\vec{b}$

$$(|\vec{a}|, 0, 0) \times (|\vec{b}|\cos\phi, |\vec{b}|\sin\phi, 0)$$

$$= (0, 0, |\vec{a}||\vec{b}|\sin\phi)$$

3) $$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & \\ & d \end{pmatrix}\left(1 + \begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix}\right)$$

$$= \det a \det d \det\left(1 + \begin{matrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{matrix}\right)$$

$$\qquad \| \quad e^{\mathrm{Tr}\log\left[\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix} + 1\right]}$$

$$\mathrm{Tr}\log\left[\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix} + 1\right] = \mathrm{Tr}\sum \frac{(-1)^{n+1}}{n}\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix}^n$$

use cyclicity of Trace $$= \mathrm{Tr}\sum \frac{(-1)^{2n+1}}{2n}\begin{pmatrix} a^{-1}b\,d^{-1}c & 0 \\ 0 & d^{-1}c\,a^{-1}b \end{pmatrix}^n$$

$$= +\frac{1}{2}\mathrm{Tr}\log\begin{pmatrix} 1 - a^{-1}b\,d^{-1}c & 0 \\ 0 & 1 - a^{-1}c\,d^{-1}b \end{pmatrix}$$

$$\Rightarrow e^{\mathrm{Tr}\log\left[\begin{pmatrix} 0 & a^{-1}b \\ d^{-1}c & 0 \end{pmatrix} + 1\right]}$$

$$= \det(1 - a^{-1}b\,d^{-1}c)$$

$$\Rightarrow \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det d \det(a - b\,d^{-1}c)$$

4) a) $H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

$$\gamma_5 H + H\gamma_5 = \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} + \begin{pmatrix} A & -B \\ C & -D \end{pmatrix} = 0$$

$$\Rightarrow \quad A = 0 \quad B = 0$$

if $\quad H\phi = \lambda\phi \quad \lambda \neq 0 \quad (\phi, \phi) = 1$

then $\quad H\gamma_5\phi = -\gamma_5 H\phi$

$$= -\lambda\gamma_5\phi$$

$\Rightarrow$ $\gamma_5\phi$ is eigenvector with eigenvalue $-\lambda$

$$(\gamma_5\phi, \gamma_5\phi) = (\phi, \phi) = 1$$

so it can be normalized

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