Topics in Quantum Many-Body Theory and Random Matrix Theory

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Abstract

1 Random Matrix Theory

1.1 Introduction

We start with the simplest random matrix theory which is the Gaussian Unitary Ensemble. The probability density of this ensemble of Hermitian $N \times N$ matrices is given by

$$P(H)DH = \mathcal{N} e^{-\frac{N\beta}{4}\mathrm{Tr}H^\dagger H} DH. \tag{1}$$

The Dyson index β denotes the number of degrees of freedom per matrix element. For a Hermitian matrix we have $\beta = 2$. For a real symmetric matrix, $\beta = 1$ and for a self-dual quaternian matrix, we have $\beta = 4$. This ensemble has a large symmetry: the probability distribution is invariant under

$$H \to UHU^{-1} \tag{2}$$

with U a $N \times N$ unitary matrix. This invariance is the reason for the simplicity and solvability of this model.

The first property is that the probability distribution of the eigenvalues and eigenvectors factorizes, i.e. if

$$H = U\Lambda U^{-1} \tag{3}$$

with Λ the diagonal matrix containing the eigenvalues, the probability distribution can be written as

$$P(H)dH = P_\Lambda d\Lambda P(U) U^{-1} dU. \tag{4}$$

To see this, we have to calculate the Jacobian from dH to $U^{-1}dU$ (the factor U^1 guarantees that the measure is invariant under $U \to VU$ and $U \to UV$. So $U^{-1}dU$ is the Haar measure.)
Excercise. Proof that the Jacoian of the transformation $\delta U \to V \delta U V^{-1}$ is 1. Hint: Write the determinant as a sum over permutations.

For the same reason, the Jacobian $dH \to U^{-1}dHU$ is also one, and

$$dH' = UdHU^{-1} = U^{-1}dU\Lambda U + d\Lambda - \Lambda U^{-1}dU. \tag{5}$$

Note that $U^{-1}dU$ is anti-Hermitian,

$$(U^{-1}dU)^\dagger = dU^{-1}U = -U^{-1}dU. \tag{6}$$

So $U^{-1}dU$ has N^2 degrees of freedom, the same number as the Hamiltonian. For $k \neq l$ we have

$$\begin{aligned} \frac{\delta H'_{kl}}{U^{-1}dU\,_{kl}} &= \Lambda_k - \Lambda_l, \\ \frac{\delta H'_{kl}}{\delta \Lambda\ _k} &= 0, \\ \frac{\delta H'_{kk}}{U^{-1}dU\,_{kl}} &= 0, \\ \frac{\delta H'_{kk}}{\delta \Lambda\ _k} &= 1. \end{aligned} \tag{7}$$

The Jacobian is this given by

$$J = \prod_{k<l} (\lambda_k - \lambda_l)^2, \tag{8}$$

resulting in the joint probability distribution of the eigenvalues

$$P(H)dH = \prod_{k<l} (\lambda_k - \lambda_l)^2 e^{-\frac{N\beta}{4} \sum_k \lambda_k^2} U^{-1} DU. \tag{9}$$

We already see that the probability to find two degenerate eigenvalues is zero, while the probability goes to zero as the square of the distance between the eigenvalues. This property is known as level repulsion. We also see that the eigenvectors are distributed uniformely over the Haar measure.

The Jacobian is known as the vanderMonde determinant. It can be written as

$$\Delta(x_k) = \prod_{k<l} (x_k - x_l) = \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \\ \vdots & & \vdots \\ x_1^{N-1} & \cdots & x_N^{N-1} \end{vmatrix}. \tag{10}$$

This allows us to add multiples of rows to a given row such that the determinant is expressed in terms of monic orthogonal polynomials with respect to the weight function $\exp(-N\beta/8x^2$. This makes it possible to perform the integration by using orthogonality relations, which is the basis of the orthogonal polynomial method. We will discuss this in more detail later in the lecture.

We will first calculate the average spectral density corresponding to the the joint probability distribution (9). There are many ways to do this, but let us determine the large N average level density from a mean field argument. We consider the eigenvalues as particles in one dimension with the eigenvalues identified as the positions of the partions. The action is thus given by

$$\frac{N}{2} \int \rho(x) x^2 - \int \int dx dy \rho(x) \rho(y) |\log(x-y)| + C \int dx \rho(x), \tag{11}$$

where we have added the last term because of the constraint that the eigenvalue density $\rho(x)$ is normalized. In the large N limit, the density is determined by the minimum of the action, and is thus given by

$$\frac{N}{2} x^2 - \int dy \rho(y) \log|x-y| = C. \tag{12}$$

By differentiation with respect to x we obtain

$$Nx = P \int \frac{dy}{x-y} \rho(y), \tag{13}$$

where the integral has to be interpreted and a principal value integral. The solution is given by

$$\rho(x) = N\pi \sqrt{2 - x^2}. \tag{14}$$

This is the famous semi-circular spectral level density of random matrix theory.

1.2 The Orthogonal Polynomial Method

Let us rescale the eigenvalues such that the joint probability distribution is given by

$$\rho(x_1, \cdots, x_N) = |\Delta(\{x_k\})|^2 e^{-N\sum_{k=1}^N V(x_k)} \tag{15}$$

The simplest cases is the GUE with $V(x) = x^2$. However, since a general potential does not complicate the expressions, we will derive our results for arbitrary $V(x)$. The Vandermonde determinant can be written as

$$\Delta(\{x_k\}) = \begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \\ \vdots & & \vdots \\ x_1^{N-1} & \cdots & x_N^{N-1} \end{vmatrix}. \tag{16}$$

By the addition of multiples of rows of lower order which do not change the determinant, each of the rows can be expressed in terms of orthogonal polynomials with respect to the weight function $\exp(-V(x))$, i.e.

$$\Delta(\{x_k\}) = \mathcal{N} \begin{vmatrix} p_0(x_1) & \cdots & p_0(x_N) \\ p_1(x_1) & \cdots & p_1(x_N) \\ \vdots & & \vdots \\ p_{N-1}(x_1) & \cdots & p_{N-1}(x_N) \end{vmatrix}. \tag{17}$$

The orthogonal polynomials are defined by

$$\int_{-\infty}^{\infty} dx e^{-NV(x)} p_k(x) p_l(x) = \delta_{kl}. \tag{18}$$

For the GUE the corresponding orthogonal polynomials are the Hermite polynomials. The normalization constant arises because in general the polynomials are not monic. It can be determined from the overall normalization of the distribution function.

It is convenient to introduce wave-functions defined by

$$\varphi_k(x) = p_k(x) e^{-NV(x)/2}. \tag{19}$$

The joint probability distribution can then be written as

$$\rho(x_1, \cdots, x_N) = \mathcal{N} \begin{vmatrix} \varphi_0(x_1) & \cdots & \varphi_{N-1}(x_1) \\ \vdots & & \vdots \\ \varphi_0(x_N) & \cdots & \varphi_{N-1}(x_N) \end{vmatrix} \times \begin{vmatrix} \varphi_0(x_1) & \cdots & \varphi_0(x_N) \\ \vdots & & \vdots \\ \varphi_{N-1}(x_1) & \cdots & \varphi_{N-1}(x_N) \end{vmatrix}. \tag{20}$$

By multiplying the two determinants, we find that joint eigenvalue distribution can be expressed as

$$\rho(x_1, \cdots, x_N) = \mathcal{N} \det_{ij}[K(x_i, x_j)] \tag{21}$$

where the kernel is defined by

$$K(x_i, x_j) = \sum_{k=0}^{N-1} \varphi_k(x_i) \varphi_k(x_j). \tag{22}$$

The kernel has the following elementary properties

$$\int K(x,x) = N, \qquad \int dy K(x,y)K(y,z) = K(x,z). \tag{23}$$

The spectral density is obtained by integrating over all eigenvalues except one. Expressing the determinant as a sum over all permutation of products of N matrix elements, one observes that in each term each argument occurs twice. Performing the integrations by means of (23) we arrive at the following expression for the spectral density (we choose the normaliztion factor such that $\int dx K(x,x) = N$)

$$\rho(x) = K(x,x). \tag{24}$$

Next we integrate over one of the eigenvalues, say x_1. Writing the determinant as a sum over permuations we obtain

$$\begin{aligned} \rho(x_2, \cdots, x_N) &= \int dx_1 \rho(x_1, \cdots, x_N) dx_1 \\ \int dx_1 \sum_\sigma (-1)^{|\sigma|} K(x_1, x_{\sigma(1)}) \cdots K(x_N, x_{\sigma(N)}). \end{aligned} \tag{25}$$

We can write the permuation as a product over cycles. If x_1 is in a cycle of length 1, we obtain N from the integral. What remains is the determinant of size $N-1$. If x_1 is in a longer cycle, we get 1 from the integral. However the sign of the permutation changes giving a factor -1, while $N-1$ cycles give the same permutation of $x_2, cdots, x_n$. What remains is (1-N) times the determinat of size $N-1$. In total we thus find the determinant of length $N-1$.

The two-point correlation function is obtained by integrating over all eigenvalues except two. In this case one finds

$$R_2(x,y) = -K(x,y)^2 + K(x,x)K(y,y). \tag{26}$$

Here, we have used that the total number of permutations for which 1 and 2 are in the same cycle is the same of the total number of permutations with 1 and 2 in different cycles. This can be seen as follows. In a cycle decomposition, a permutation with 1 and 2 in the same cycle can be written as $(1p2q)\pi_r$ or $(1q2p)\pi_r$ for a sequence of numbers p and q and remaining cycles π_r. For 1 and 2 in different cycles we have the permutations $(1p)(2q)\pi_r$ and $(1q)(2p)\pi_r$. From the length of the cycles one immediately finds that the signature of the two groups of permutations is opposite. It is also easy to see that the numerical factor for given p, q and π_r is the same is the two groups. This result determines (26) up to a overall constant. We observe that the correlation function vanishes for $x = y$ which also follows from the determinantal structure of the correlation function.

To find the proportionality constant we notice that

$$\int dx R_1(x) dx = N, \tag{27}$$

$$\int dx dy R_2(x,y) = N(N-1), \tag{28}$$

where the factor $(N-1)$ arise because all eigenvalues in the joint probability distribution are necessarily different. The proportionality constant is thus as given in (26).

2 The Resolvent

The resolvent plays an important role in random matrix theory. It is defined as

$$G(z) = \left\langle \mathrm{Tr} \frac{1}{z - H} \right\rangle, \tag{29}$$

where the brackets denote the average over the ensemble. For large z it can be expanded in a geometric series as

$$G(z) = \sum_k \frac{\langle \mathrm{Tr} H^k \rangle}{z^{k+1}}. \tag{30}$$

The $\langle \mathrm{Tr} H^k \rangle$ are known as the moments of the average eigenvalue density.

In terms of eigenvalues of H, the resolvent is given by

$$G(z) = \left\langle \sum_k \frac{1}{z - \lambda_k} \right\rangle. \tag{31}$$

It is clear that $G(z)$ is an analytic function of z except on the locus of the eigenvalues. The spectral density is related to the resolvent as

$$rho(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} [G(x + i\epsilon) - G(x - i/epsilon)] \tag{32}$$

This follows immediately from

$$\int dx \frac{\epsilon}{(x - \lambda)^2 + \epsilon^2} = \delta(x - \lambda). \tag{33}$$

2.1 Generalized Resolvents

Eynard [**?**] introduced generalized resolvents

$$\bar{W}_k(x_1, \cdots, x_k) = N^{k-2} \left\langle \mathrm{tr} \frac{1}{x_1 - M} \frac{1}{x_2 - M} \cdots \frac{1}{x_k - M} \right\rangle. \tag{34}$$

Topological expansion

$$\bar{W}_k(x_1, \cdots, x_k) = \sum_{h=0}^{\infty} N^{-2h} W_k^{(h)}(x_1, \cdots, x_k). \tag{35}$$

2.2 Spectral Correlation Function

The two point correlation function is defined by

$$\rho_2(\lambda, \lambda') = \langle \rho(\lambda) \rho(\lambda') \rangle - \langle \rho(\lambda) \rangle \langle \rho(\lambda') \rangle. \tag{36}$$

The correlation function $\rho_2(\lambda, \lambda')$ includes a term in which the eigenvalues are equal, and can thus be decomposed as

$$\rho_2(\lambda, \lambda') = \delta(\lambda - \lambda') \langle \rho(\lambda) \rangle + R_2(\lambda, \lambda'). \tag{37}$$

The two-point correlation function satisfies the sum rule

$$\int d\lambda \rho_2(\lambda, \lambda') = 0, \tag{38}$$

where the integral is over the complete spectrum.

2.3 Unfolding

One frequently employs the integrated spectral density

$$N(E) = \int_{-\infty}^{E} \rho(\lambda) d\lambda. \tag{39}$$

Since $N(E)$ jumps by one at the position of each eigenvalue, it is also know as the staircase function,

Generally, the spectral density can be smoothened over a scale that is much larger than the average level spacing. Let us denote the smoothened spectral density by $\bar{\rho}$. Typically, $\bar{\rho}$ depends on the specific properties of the system. Therefore one would like to eliminate this dependence from the spectrum. This is achieved by the so-called unfolding procedure. The unfolded spectrum is given by

$$\lambda_k^{\rm unf} = \int_0^{\lambda_k} \bar{\rho}(\lambda) d\lambda. \tag{40}$$

One can easily verify that the average spacing of the unfolded sequence is equal to unity. To compare the statistical properties of spectra to random matrix theory, it is essential to use unfolded eigenvalues.

2.4 Number Variance

A convenient way to study the correlations of eigenvalues is to use the number variance. It is the variance of the number of eigenvalues in a interval ΔE containing $\bar{n}$ eigenvalues on average. The actual number of levels in the interval is given by

$$n = \int_{\Delta E} \rho(x) dx, \tag{41}$$

so that

$$\bar{n} = \int_{\Delta E} \langle \rho(x) \rangle dx. \tag{42}$$

The number variance is given by

$$\begin{aligned} \Sigma^2(\bar{n}) &= \langle (\int_{\Delta E} \langle \rho(x) \rangle dx)^2 \rangle - \langle \int_{\Delta E} \langle \rho(x) \rangle dx \rangle^2 \\ &= \int_{\Delta E} dx \int_{\Delta E} dy \left[\langle \rho(x)\rho(y) \rangle \langle \rho(x) \rangle \langle \rho(y) \rangle \right] \\ \int_{\Delta E} dx \int_{\Delta E} R_2(x,y). \end{aligned} \tag{43}$$

We can easlily calculate the number variance for the first term in (37)

$$\Sigma^2(\bar{n}) = \int_{\Delta E} \langle \rho(x) dx = n \tag{44}$$

This is the result for the number variance in case the eigenvalues are uncorrelated.

In the literature one also frequently uses the quantity $Y_2(\lambda, \lambda')$ for the two-point correlation function of the unfolded eigenvalues

$$Y_2(\lambda, \lambda') = -R_2(\lambda, \lambda'), \tag{45}$$

where the minus sign is conventional. If a smoothened average spectral density can be defined, it is natural to expect that Y_2 is translational invariant, i.e.,

$$Y_2(\lambda, \lambda') = Y_2(\lambda - \lambda'). \tag{46}$$

In that case, the number variance can be expressed as

$$\Sigma^2(L) = L - \int_0^L (L - r) Y_2(r) dr. \tag{47}$$

2.5 The Spectral Form Factor

The spectral form factor is the Fourier transform of the two-point correlation function

$$K(\tau) = \int dx \int dy e^{i\tau(x-y)} R_2(x, y). \tag{48}$$

For the first term in (37) we obtain

$$K(\tau) = \int dx \langle \rho(x) \rangle = N. \tag{49}$$

Because of the sum-rule (38), we have that

$$K(0) = 0. \tag{50}$$

Because of the oscillatory exponent, the spectral form factor is very noisy which can be improved by including a Gaussian cutoff $\sim \exp(-(x-y)^2/2/w^2)$.

3 Average Spectral Density and Two-Point Correlator for the GUE

In case of the GUE the potential is given by $V(x) = x^2$. The corresponding orthogonal polynomials are the Hermite polynomials. The correctly normalized wave functions are given by

$$\frac{N^{1/4}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-N\frac{x^2}{2}}. \tag{51}$$

The spectral density is thus given by

$$\rho(x) = \sum_{n=0}^{N-1} \frac{\sqrt{N}}{2^n \sqrt{\pi} n!} H_n(x\sqrt{N}) H_n(x\sqrt{N}) e^{-Nx^2}. \tag{52}$$

This sum can be calculated exactly by means of the Christoffel-Darboux formula

$$\sum_{n=0}^{N-1} \frac{H_n(y) H_n(y)}{2^n n!} = \frac{1}{2^N (N-1)!} [H_N(y) H_N(y) - H_{N-1}(y) H_{N+1}(y)]. \tag{53}$$

In the limit $N \to \infty$ we can use the asymptotic limit of the Hermite polynomials given by [**?**]

$$\frac{N^{1/4} H_{N-k}(x\sqrt{N}) e^{-N\frac{x^2}{2}}}{(2^{N-k}(N-k)!\sqrt{\pi})^{1/2}} = \sqrt{\frac{\sqrt{2}}{\pi \sin\phi}} \cos(N(\phi - \sin(2\phi)/2) - (k-\frac{1}{2})\phi - \frac{\pi}{4}) + O(1/N). \tag{54}$$

Here, $\cos\phi = x/\sqrt{2}$. This asymptotic result is valid for $N \to infty$ and k finite. If we use that

$$\cos\alpha\cos\alpha - \cos(\alpha+\beta)\cos(\alpha-\beta) = \sin^2\beta, \tag{55}$$

we find for the spectral density to $O(1/N)$

$$\rho(x) = \frac{N\sqrt{2}}{\pi\sin\phi}\sin^2\phi = \frac{N}{\pi}(2-x^2)^{1/2}. \tag{56}$$

We observe that the average spectral density has the shape of a semicircle, and is correctly normalized to N.

Next we evaluate the two-point correlation function. In this case we use the Christoffel-Darboux formula for different values of the argument

$$\sum_{n=0}^{N-1} \frac{H_n(x)H_n(y)}{2^n n!} = \frac{1}{2^N(N-1)!}\frac{H_N(x)H_{N-1}(y) - H_{N-1}(y)H_N(x)}{x-y}. \tag{57}$$

We study the spectral correlation function in the microscopic limit, i.e. in the limit $N \to \infty$ with $(x-y)N$ fixed. Keeping only terms to leading order in $1/N$ we find for the kernel

$$\begin{aligned} K(x,y) = \quad & \frac{1}{\pi\sin\phi}[\cos(N\zeta(\phi) + \phi/2 - \frac{\pi}{4})\cos(N\zeta(\phi') - \phi/2 - \frac{\pi}{4}) \\ - \quad & \cos(N\zeta(\phi') + \phi/2 - \frac{\pi}{4})\cos(N\zeta(\phi) - \phi/2 - \frac{\pi}{4})], \end{aligned} \tag{58}$$

where

$$\zeta(\phi) = \phi - \sin(2\phi)/2, \tag{59}$$

$$\cos\phi = x/\sqrt{2}, \tag{60}$$

$$\cos\phi' = y/\sqrt{2}. \tag{61}$$

Using an addition formula for the goniometric functions we find

$$K(x,y) = \frac{1}{\pi\sin\phi}\frac{\sin(N\zeta(\phi') - N\zeta(\phi))\sin\phi}{x-y}. \tag{62}$$

By inspection we find that the function $\zeta(\phi)$ satisfies

$$\frac{d\zeta}{dx} = -\pi\rho(x)/N, \tag{63}$$

where

$$\rho(x) = \frac{N}{\pi}\sqrt{2-x^2}. \tag{64}$$

In the microscopic limit x and y are close and we can make the approximation

$$N\zeta(\phi') = N\zeta(\phi) + N(y-x)\frac{d\zeta}{dx}. \tag{65}$$

For the kernel we thus find

$$K(x,y) = \frac{1}{\pi}\frac{\sin\pi(x-y)\rho(x)}{x-y} \tag{66}$$

The unfolded distance between two eigenvalues is given by

$$r = (x-y)\rho(x). \tag{67}$$

This results in the unfolded two-point correlation function

$$Y_2(r) = \frac{sin^2\pi r}{\pi^2 r^2}. \tag{68}$$

The number variance is then given by

$$\Sigma^2(n) = n - 2\int_0^n (n-r)Y_2(r). \tag{69}$$

The first terms of the r.h.s. cancels because $\int_0^\infty dr Y_2(r) = 1/2$. The remaining integral is given by

$$\Sigma_2(n) = \frac{1}{\pi^2}(\log(2\pi n) + \gamma + 1), \tag{70}$$

where the logarithmic dependence follows from the asymptotic behavior of $Y_2(r)$ (with $\sin^2(\pi r)$ replaced by its average).

Excercise 3.2. Show this result from the expression for the number variance given in lecture 1.

4 Classification of Random Matrix Theories

4.1 Unitary versus Anti-Unitary Symmetries

A symmetry in quantum-mechanics preserves the absolute value all scalar products of the system. If U is a symmetry, then

$$|\langle U\psi|U\phi\rangle| = |\langle\psi|\phi\rangle|. \tag{71}$$

This give two possibilities

- $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle$. In this case U is unitary.
- $\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle^*$ In this case U is anti-unitary.

This is sometimes referred to a Wigner's fundamental theorem [8].

For a Unitary symmetry U,

$$[U, H] = 0, \tag{72}$$

we can find a set of basis functions that diagonalize both H and and a commuting subset of the generators G_i of $U = \exp \alpha_k G_g$,

$$H\phi_{kj_i} = E_k \phi_{kj_i}, \qquad G_i \phi_{kj_i} = g_{j_i} \phi_{kj_i}, i = 1, \cdots, p. \tag{73}$$

In this basis, the Hamiltonian splits into blocks labeled by the quantum numbers g_{j_i}. In random matrix theory it is always assumed that the Unitary symmetries have been accounted exactly, and we are considering an irreducible block of the Hamiltonian. And example is to consider all states of a given spin and parity.

This leaves us with with the anti-unitarty symmetry. The best known anti-unitary symmetry is time-reversal invariance, T. For integer spin this operators is given by complex conjugation operator,

$$T = K, \tag{74}$$

while for half-integer spin, it is equal to

$$T = K\sigma_2. \tag{75}$$

A fundamental difference between these two cases is that

$$K^2 = 1, \qquad (K\sigma_2)^2 = -1. \tag{76}$$

More generally an anti-unitary symmetry operator is given by AK with A a unitary symmetry. We have that $(AK)^2 = AA^*$ is unitary, and since we took care of the unitray symmetries, we have that $(A)^2 = \lambda \mathbb{I}$

The value of the Dyson index is determined by the anti-unitary symmetries of the system. If there are no anti-unitary symmetries the Hamiltonian is Hermitean and the value of $\beta = 2$. In case there are anti-unitary symmetries given by

$$[KA, H] = 0, \tag{77}$$

with K a complex conjugation operator and A is an unitary matrix, we have to distinguish two cases:

$$(KA)^2 = 1 \quad \text{and} \quad (KA)^2 = -1, \tag{78}$$

corresponding to $\beta = 1$ and $\beta = 4$, respectively. We now show that these are the only two possibilities.

Since KA is anti-unitary, we have that $(KA)^2$ is unitary, and because all unitary symmetries have already been taken into accoount, we necessarily have that

$$(KA)^2 = A^*A = \lambda \mathbb{I} \qquad \text{and} \qquad |\lambda| = 1. \tag{79}$$

Therefore,

$$[A, A^*] = [A, \lambda A^{-1}] = 0, \tag{80}$$

so that

$$\lambda^* \mathbb{I} = (A^* A)^* = A A^* = A^* A = \lambda \mathbb{I}, \tag{81}$$

and $\lambda = 1$ or $\lambda = -1$.

In the first case it is always possible to find a basis in which the Hamiltonian is real. Starting basis vector ϕ_1 we construct $\psi_1 = \phi_1 + KA\phi_1$. Then choose ϕ_2 perpendicular to ψ_1 and define $\psi_2 = \phi_1 + KA\phi_1$. Then

$$\begin{aligned} & (\phi_2 + KA\phi_2, \psi_1) \\ = \; & (KA\phi_2, \psi_1) \\ = \; & ((KA)^2\phi_2, KA\psi_1)^* \\ = \; & (\phi_2, \psi_1) = 0. \end{aligned} \tag{82}$$

The next basis vector is found by choosing ϕ_3 perpendicular to ψ_1 and ψ_2, etc.

In this base the Hamiltonian is real

$$\begin{aligned} H_{kl} = (\psi_k, H\psi_l) & \\ = \; & (KA\psi_k, KAH\psi_l)^* \\ = \; & (\psi_k, HKA\psi_l)^* \\ = \; & (\psi_k, H\psi_l)^* = H^*_{kl}. \end{aligned} \tag{83}$$

The best know anti-unitary operator in this class is the time-reversal operator. In that case A is the identity matrix.

In the case $(KA)^2 = -1$ all eigenvalues of the Hamiltonian are doubly degenerate. This can be show as follows. If ϕ_k is and eigenvector with eigenvalue λ_k, then it follows from the commutation relations that also $KA\phi_k$ is an eigenvector of the Hamiltonian with the same eigenvalue. The important thing is that this eigenvector is perpendicular to ϕ_k,

$$(\phi_k, KA\phi_k) = (KA\phi_k, (KA)^2\phi_k)^* = -(\phi_k, KA\phi_k). \tag{84}$$

One can prove that in this case it is possible to chose a basis in which the Hamiltonian matrix can be organized into real quaternions [**?**]. The eigenvalues of a Hermitean quaternion real matrix are quaternion scalars, and the eigenvalues of the original matrix are thus doubly degenerate in agreement with (84). The best known example of this Kramers degeneracy is for time reversal invariant systems of half-integral spin but no rotational invariance. Then the time reversal operator is given by $K\sigma_2$ with $(K\sigma_2)^2 = -1$.

4.2 Classification of Random Matrix Theories

If we have no other constraints, we have three different possibilities for the Gaussian Random Matrix theories with probability distribution

$$P(H)dH = e^{-\frac{N\beta}{4D}\mathrm{Tr}H^2} \tag{85}$$

- There are no anti-unitary symmetries. Then the matrix elements of H are complex, and the probability distribution is invariant under $H \to UHU^{-1}$ with $U^\dagger U = 1$. This ensemble is known as the Gaussian Unitary Ensemble (GUE) with Dyson index $\beta_D = 2$.

- The case $(KA)^2 = 1$. Then the Hamiltonian is real, and the ensemble is invariant under orthogonal transformation $H \to O^T HO$ with $O^T O = 1$. This ensemble is known as the Gaussian Orthogonal Ensemble (GOE) with Dyson index $\beta_D = 1$.

- The case $(KA)^2 = -1$. Then the Hamiltonian is quaternion real or self-dual qauternion, i.e

$$H^T = \tau_2 H \tau_2. \tag{86}$$

with τ_2 a direct product of 2×2 Pauli matrices σ_2 So we obtain from $H^\dagger = H$

$$H_{kl} = H^* lk = q^*_{0,lk} - i\sigma_k q^*_{0,lk} \tag{87}$$

and from the self-duality relation

$$H_{kl} = \tau_2 H_{lk} \tau_2 q_{0,lk} - i\sigma_k q_{0,lk} \tag{88}$$

so that $q_k \in \mathcal{R}$. The probability distribution is invariant under unitary transformation with

$$U^T \tau_2 U = \tau_2, \tag{89}$$

which are the symplectic unitary matrices. This ensemble is known as the Gaussian Symplectic Ensemble with Dyson index $\beta_D = 4$.

4.3 Involutive Symmetries

x In addition to anti-unitary symmetries, the ensemble may also have involutive symmetries

$$I(IHI^{-1})I^{-1} = H. \tag{90}$$

The first possibility is that anti-unitary transformation is an involution,

$$AKH(AK)^(- 1) = -H. \tag{91}$$

If $(AK)^2 = 1$, then A can be absorbed in the ensemble so that

$$H^* = -H, \tag{92}$$

and H is purely imaginary. Because H is also Hermitian, it must be anti-symmetric. The eigenvalues of this ensemble of random matrices occur in pairs $\pm\lambda$ for even N, while for odd N there is one zero eigenvalue and all other eigenvalues also occur in pairs $\pm\lambda$. In the Cartan classification, this is the ensemble D.

The second possibility is $(AK)^2 = -1$. Up to unitary transformations we then obtain

$$\sigma_2 H^* \sigma_2 = -H. \tag{93}$$

This is the ensemnble of anti-selfdual quaternion matrices. It eigenvalues come in pairs $\pm\lambda$: If

$$H\phi = \lambda\phi, \tag{94}$$

then

$$H\sigma_2\phi^* = -\sigma_2 H^*\phi^* = -\lambda\sigma_2\phi. \tag{95}$$

In the Cartan classification, this is the ensemble C.

4.4 The Chiral Ensemble

The second involution we are considering is

$$\Gamma_5 H\Gamma_5 = -H. \tag{96}$$

This is the chiral symmetry of the Dirac operator for even-dimensional matrices. $Gamma_5$ is the diagonal matrix with its first half diagonal elements equal to $+1$ and the remaining ones equal to -1. A direct consequence of this symmetry is the H has the block structure

$$H = \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix}. \tag{97}$$

It eigenvalues are double degenerate. If

$$H\phi = \lambda\phi, \tag{98}$$

then

$$H\Gamma_5\phi = -\Gamma_5 H\phi = -\lambda\Gamma_5\phi. \tag{99}$$

If may happen that $\Gamma_5\phi \sim \phi$, then necessarily $\lambda = 0$. These are the zero modes of the Dirac operator.

If there are no anti-unitary symmetries, the matrix elements are arbitry complex. Such a matrix can be "diagonalized" by a principal value decomposition

$$W = U\Lambda V^1, \tag{100}$$

with U and V unitary matrices and Λ a diagonal semi-positive definite matrix. This ensemble is known as the chiral Gaussian Unitary Ensemble (chGUE).

For the Dirac operator, the charge conjugation matrix for dimensions d with even $d/2$, is an anti-unitary operator that commutes with the Dirac operator. The gauge field also may have reality conditions, in the adjoint representation, they are real for any number of colors, and $SU(2)$ in the fundamental representation is also real. We have two posibilites in case $AK, \Gamma_5] = 0$,

$$(AK)^2 = 1, \qquad \text{and} \qquad (AK)^2 = -1. \tag{101}$$

In the first case the off-diagonal block of H is real. This ensemble is invariant under $W \to O_1 W O_2$ with O_1 and O_2 orthogonal matrices, and is know as the chiral Gaussian Orthogonal

Ensemble (chGOE). In the second case, we can always find a basis in which the matrix elements of W are quaternion real. This ensemble is invariant under $W \to S_1 W S_2$, with S_1 and S_2 symplectic matrices, and is know and the chiral Gaussian Symplectic Ensemble (chGSE).

The second case is when the anti-unitary symmetry operater does not commute with Γ_5. We still insist on the block-form (97) but can no longer conclude that the matrix elements of W are real or quaternion real. Let us consider the anti-unitary operater $\Sigma_1 K$ which anti-commutes with Γ_5. From the commutation relation with H we obtain

$$\begin{aligned} \Sigma_1 K H \Sigma_1 K &= \Sigma_1 K \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} \Sigma_1 K \\ &= \Sigma_1 \begin{pmatrix} 0 & W^* \\ W^T & 0 \end{pmatrix} \Sigma_1 \\ &= \begin{pmatrix} 0 & W^T \\ W^* & 0 \end{pmatrix}. \end{aligned} \tag{102}$$

We thus find that $W^T = W$ which is a complex symmetric matrix. In the Cartan classification, this is the ensemble CI.

The second case is the anti-unitary operator $i\Sigma_2 K$ which squares to -1. Then we find

$$\begin{aligned} i\Sigma_2 K H (i\Sigma_2 K)^{-1} &= i\Sigma_2 K \begin{pmatrix} 0 & W \\ W^\dagger & 0 \end{pmatrix} (-i\Sigma_2 K) \\ &= -i\Sigma_2 \begin{pmatrix} 0 & W^* \\ W^T & 0 \end{pmatrix} i\Sigma_2 \\ &= \begin{pmatrix} 0 & -W^T \\ -W^* & 0 \end{pmatrix}. \end{aligned} \tag{103}$$

This implies that $W^T = -W$, so that W is a complex anti-symmetric matrix. In the Cartan classification, this is the random matrix theory DIII

4.5 Classification of the Random Matrix Ensembles in terms of Symmetric Spaces

The Random Matrix Ensembles in this section were classified according to the Cartan classification of symmetric spaces. A symmetric space is a manifold such that every point is a fixed point of an involutive isometry (i.e. $x_i \to -x_i$). The Riemann curvature tensor is covariantly constant in a symmetric space. A symmetric space is best characterized via the notion of symmetric pair. A symmetric pair (G, H) is defined as a pair of a connected Lie group G and a closed subgroup H such that an involutive analytic authomorphism σ of G exists with $H \in H_\sigma$, where H_σ is the set of fixed points of σ. Then, with some additional conditions (see the book by Helgason [**?**] for more details) the coset G/H is a symmetric space.

As an example, consider the group $U(p+q)$ and define Γ_{pq} as the diagonal matrix with the first p diagonal matrix elements equal to 1 and the remaining q diagonal matrix elements equal to -1. Consider the transformation

$$\sigma(g) = \Gamma_{pq} g \Gamma_{pq}, \tag{104}$$

which is an involution because obviously $\sigma(\sigma(g)) = g$. One can also easily show that it is an analytic automorphism. The set of fixed points of σ is the group $U(p) \times U(q)$. Therefore, $U(p+q)/U(p \times U(q)$ is a Riemannian symmetric space.

A symmetric space can be of the compact, noncompact or the Euclidean type with positive, negative or zero curvature, respectively. Each of the random matrix ensembles discussed in this section is tangent to one of the large classes of symmetric spaces. The complete classification is given in table I [**?**] where only the symmetric space of the compact type is given

Table 1: Random matrix ensemble, corresponding symmetric space, and the value for β.

RMT	symmetric space	compact Lie-group structure	β	Matrix Tyoe
GOE	AI	$U(N)/O(N)$	1	Real Symmetric
GUE	A	$U(N)$	2	Hermitian
GSE	AII	$U(2N)/Sp(N)$	4	Quaternion Real
chGOE	BDI	$SO(p+q)/SO(p) \times SO(q)$	1	Chiral Real
chGUE	AIII	$U(p+q)/U(p) \times U(q)$	2	Chiral Complex
chGSE	CII	$Sp(p+q)/Sp(p) \times Sp(q)$	4	Chiral Quanternion Real
AZ-CI	CI	$Sp(N)/U(N)$	1	Chiral Complex Symmetric
AZ-D	D	$SO(N)$	2	Imaginary Anti-Symmetric
AZ-C	C	$Sp(N)$	2	Anto-Selfdual Quaternion
AZ-DIII	DIII	$SO(2N)/U(N)$	4	Chiral Complex Anti-Symmetruc

For example, for special unitary matrices we can write $U = 1 + iH + \cdots$. Therefore, the Hermitean matrices are tangent to the space A (after dividing out a $U(1)$ factor). As another example, the generators of the class AIII are given by matrices with the structure (**??**).

5 The Replica Trick

The replica trik is widely used in random matrix theory as well as in the SYK model. There are two main applications first, the calculation of the free energy of a disordered system

$$\log Z = \lim_{n \to 0} \frac{Z^n - 1}{n}, \tag{105}$$

and second to to calculate the quenched averages,

$$G(z) = \mathrm{Tr} \frac{1}{z + H} \lim_{n \to 0} \frac{1}{n} \frac{d}{dz} \det{}^n (H + z). \tag{106}$$

The rationale is that it is hard to calculate disorder averages of $\log Z$ or quenched averages, while it is much easier to calculate the replicated partition function Z^n or the average of a spectral determinant. The idea is to calculate the average for integer values of n and then analytically continue in n and take the limit $n \to 0$. According to Carlson's theorem, if an analytic function f is zero for positive integers n, then it is identically equal to zero under some conditions. Clearly $f(n)$ is not unique. For example $\sin \pi n$ vanishes for integer, and the conditions of the theorem are to exclude such functions.

There is another way the replica trick can work, namely if there is an recursion relation relating the Z_n. Then we can take the limit $n \to 0$ of the recursion relation to find the resolvent [**?**].

Generally, the replica trick works for mean field calculations, and it is assumed that the mean field solution is proportional to the identity in replica space.

5.1 Replica Calculation of the Resolvent for the GUE

The fermion determinant can be represented as a Grassmann integral

$$\langle \det{}^n(z+H)\rangle = \int dH e^{-\frac{N}{2}TrH^2} \int d\psi d\bar{\psi} e^{\sum_{k=1}^n \bar{\psi}_i^k (z+H)_{ij} \psi_j^k}. \tag{107}$$

Because the Hamiltonian is Hermitian we only integrate over H_{ij} with $i < j$. The diagonal matrix elements have to be treated separately, but one can check that we get the right answers by only considering the case $i < J$. The exponent becomes

$$\begin{aligned} & -N\sum_{i<j} H_{ij}H_{ji} + \sum_{k,l=1}^n \bar{\psi}_i^k H_{ij}\psi_j^k + \bar{\psi}_j^l H_{ji}\psi_j^l \\ = \quad & -N\sum_{i<j}(H_{ij} - \frac{1}{N}\sum_{k=1}^n \bar{\psi}_i^k\psi_j^k)(H_{ji} - -\frac{1}{N}\sum_{l=1}^n \bar{\psi}_j^l\psi_i^l) + \sum_{i<j}\frac{1}{N}\sum_{k=1}^n \bar{\psi}_i^k\psi_j^k \sum_{l=1}^n \bar{\psi}_j^l\psi_i^l. \end{aligned} \tag{108}$$

We can now shift the integration variables

$$H_{ij} \to H_{ij} + \sum_{k,l=1}^n \bar{\psi}_i^k H_{ij}\psi_j^k, \tag{109}$$

and then do the Gaussian integral over H_{ij} which just gives a constant. What remains is the last term in eq. (108) which can be rewritten as

$$-\frac{1}{2N}\sum_{k=1}^n\sum_{l=1}^n\sum_{i=1}^n \bar{\psi}_i^k\psi_i^l \sum_{j=1}^n \bar{\psi}_j^l\psi_j^k. \tag{110}$$

There is a factor $1/2$ because we extended the sum from$i < j$ to all i and j. Next we consider the integral

$$\int d\sigma e^{\frac{N}{2}(i\sigma^{kl} + \frac{1}{N}\bar{\psi}^k\cdot\psi^l)(i\sigma^{lk} + \frac{1}{N}\bar{\psi}^l\cdot\psi^k)}. \tag{111}$$

Obviously, the value of this integral is a constant. If we expand the exponent and bring the fourth order fermion term to the other side, we see that (110) can be written as

$$\int d\sigma e^{\frac{N}{2}i\sigma^{kl}i\sigma^{lk} + \frac{1}{2}i\sigma_{lk}\bar{\psi}^k\cdot\psi^l + \frac{1}{2}i\sigma_{kl}\bar{\psi}^l\cdot\psi^k}. \tag{112}$$

If we take $\bar{\psi}^k \equiv \psi^{k^*}$ and use complex conjugation of the second kind

$$(\psi^*)^* = -\psi^*, \tag{113}$$

we find that

$$(\bar{\psi}^k \cdot \psi^l)^* = \bar{\psi}^l \cdot \psi^k, \tag{114}$$

so that we can choose σ Hermitian. Therefor the two last terms in (**??**) are equal and we finally obtain

$$\int d\sigma e^{\frac{N}{2} i\sigma^{kl} i\sigma^{lk} + i\sigma^{lk} \bar{\psi}^k \cdot \psi^l}. \tag{115}$$

Now the integral over the Grassmann variables can be performed trivially resulting in

$$\int d\sigma e^{\frac{N}{2}(i\sigma^{kl}) i\sigma^{lk} + N \mathrm{Tr} \log(i\sigma + z)}.. \tag{116}$$

We have that

$$\frac{d}{dz} = \sum_k \frac{d}{d i\sigma_{kk}}. \tag{117}$$

After a partial integration, we thus find that the resolvent is given by

$$G(z) = \frac{1}{n} \sum_k (-i\sigma_{kk}). \tag{118}$$

Next we calculate the integral over σ by a saddle-point approximation. The saddle-point equation is given by

$$i\sigma_{kl} + [1/(i\sigma + z)]_{kl} = 0. \tag{119}$$

For the solution We use the ansatz

$$\bar{\sigma}_{kl} = \bar{\sigma} \delta_{kl} \tag{120}$$

resulting in

$$i\bar{\sigma} + \frac{1}{i\bar{\sigma} + z} = 0. \tag{121}$$

The solution is given by

$$i\bar{\sigma} = -\frac{z}{2} \pm \frac{1}{2}\sqrt{z^2 - 4} \tag{122}$$

This results in the resolvent

$$G(z) = \frac{z}{2} \mp \frac{1}{2}\sqrt{z^2 - 4}, \tag{123}$$

Where we have trivially taken the replica limit. Since the resolvent has to go as $1/z$ for $z \to \infty$ we have to choose the solution with the minus sign. The level density is given by

$$\rho(x) = -\frac{1}{\pi} \Im G(x + i\epsilon) = \frac{1}{2\pi}\sqrt{x^2 - 4}. \tag{124}$$

(there is a minus sign because we started with $1/(z + H)$).

5.2 Loop Expansion

We write

$$i\sigma = i\bar{\sigma} + i\alpha. \tag{125}$$

Then

$$G(z) = \bar{G}(z) + \langle i\alpha \rangle \tag{126}$$

with

$$\begin{aligned} \langle i\alpha \rangle &= \lim_{n\to 0} \frac{1}{n} \int d\alpha e^{-\frac{N}{2}\alpha^2 + N\mathrm{Tr} i\alpha i\bar{\sigma} + N\mathrm{Tr}\log(1-i\bar{\sigma}i\alpha)} \\ &= \lim_{n\to 0} \frac{1}{n} \int d\alpha e^{-\frac{N}{2}(1-\bar{\sigma}^2)\alpha^2 + N\mathrm{Tr} i\alpha i\bar{\sigma} + N\sum_{k\geq 3}\frac{1}{k}(-1)^{k+1}\mathrm{Tr}(\bar{G}(z)i\alpha)^k}. \end{aligned} \tag{127}$$

The propagator is thus given by

$$\alpha^{kl}\alpha^{mn} = \frac{\delta_{kn}\delta_{ml}}{N(1-\bar{G}(z)^2)} \tag{128}$$

with vertices

$$-\frac{1}{k}\mathrm{Tr}(\bar{G}(z)i\alpha)^k \tag{129}$$

.

The 1/N correction is given by

$$-\frac{1}{N}\lim_{n\to 0}\frac{1}{n}\bar{G}^3(z)\frac{1}{(1-\bar{G}^2(z))^2}n^2, \tag{130}$$

and thus vanishes in the replica limit. In fact all corrections $1/N^{2k+1}$ vanish, and that is why this expansion is callled a topological expansion.

Let us now evaluate the $1/N^2$ correction

$$\begin{aligned} & -N\frac{1}{n}\frac{1}{5}\bar{G}^5(z)\langle \mathrm{Tr} i\alpha \mathrm{Tr}(i\alpha)^5\rangle \\ = \quad & \frac{1}{N^2}\frac{\bar{G}^5(z)}{(1-\bar{G}^2(z))^3}. \end{aligned} \tag{131}$$

The crossed contraction in the five-point vertex allows only one free summation index resulting in a finite replica limit.

5.3 Two-Point Function

The two-point function is given by

$$\langle G(x)G(y)\rangle = \lim_{n\to 0}\frac{1}{n^2}\frac{1}{N^2}\frac{d}{dx}\frac{d}{dy}\det{}^n(H+x+i\epsilon)\det{}^n(H+y-i\epsilon) \tag{132}$$

The generating function can be written as

$$\det{}^n(H+x+i\epsilon)\det{}^n(H+y-i\epsilon) = \int d\psi d\bar{\psi} e^{\bar{\psi}^{k,1}(H+x+i\epsilon)\psi^{k,1} + \bar{\psi}^{k,2}(H+y-i\epsilon)\psi^{k,2}} \tag{133}$$

This partition function has an $U(2n)$ and we will see that it is spontaneously broken to $U(n) \times U(n)$ in the limit $N \to \infty$.

Going through the same steps as for the one-point function, we obtain the σ model.

$$\int d\sigma e^{\frac{N}{2}(i\sigma_{kl}^{pq})i\sigma_{lk}^{pq}+N\mathrm{Tr}\log(i\sigma+z)}. \tag{134}$$

It looks exactly the same as in the case of the one-point function, but σ now has the block form

$$\sigma = \begin{pmatrix} \sigma_{kl}^{11} & \sigma_{kl}^{12} \\ \sigma_{kl}^{21} & \sigma_{kl}^{22} \end{pmatrix} \tag{135}$$

and

$$z = (x, \cdots, x, y, \cdots, y). \tag{136}$$

Since σ is Hermitian, it can be diagonalized

$$\sigma = U\Lambda U^{-1}. \tag{137}$$

The results in the partition function

$$Z(x,y) \int dU d\Lambda \prod_{k<l} (\Lambda_k - \Lambda_l)^2 e^{-\frac{N}{2}\sum_{k=1}^{2n}\Lambda_k^2 + NTr\log(i\Lambda+\bar{z}+U(z-\bar{z})U^{-1})}, \tag{138}$$

where $\bar{z} = (x+y)/2$. We now do the Λ integral by a saddle-point approximation in the case that $x-y = O(1/N)$ so that the last term can be neglected in the saddle-point calculation. The saddle point equation is given by

$$i\Lambda = -\frac{1}{i\Lambda + \bar{z}}, \tag{139}$$

which is solved by

$$i\Lambda = -\frac{z}{2} \pm \sqrt{\bar{z}^2 - 4}. \tag{140}$$

Becaue of the Jacobian, the leading contribution is obtained by taking half the solutions with a $+$ sign and the other half with a $-$ sign. This is consitent with the choice of $i\epsilon$.

For simplicity, let us look at the center of the spectrum, $\bar{z} = 0$. Then the resolvent is given by

$$G(\bar{z}) = (i, \cdots, i, -i, \cdots, -i). \tag{141}$$

Therefore the $U(2n)$ symmetry is spontaneously broken to $U(n) \times U(n)$. If we expand the partition function to first order in $z\bar{z}$ we obtain the partition function for the Goldstone modes

$$Z(x,y) \int_{U(2n)/U(n)\times U(n)} e^{iN\mathrm{Tr}(z-\bar{z})\Gamma_5 U \Gamma_5 U^{-1}}. \tag{142}$$

The integral is convergent because of the $i\epsilon$ in $z - \bar{z}$. This is the leading term of the chiral Lagrangian for QCD in 3 dimensions. It is an Itzykson-Zuber integral which can be evaluated analytically [**?**]. However, to do that we have to lift the degeneracies of Γ_5, which makes the

calculattion somewhat more involved. It is more instructive to calculate the integrald perturbatively. The matrix U can be parameterized as

$$U = e^{\begin{pmatrix} 0 & iW \\ -W^\dagger & 0 \end{pmatrix}} \tag{143}$$

with W a complex $n \times n$ matrix. We have that

$$\Gamma_5 U^{-1} = U\Gamma_5. \tag{144}$$

We thus find

$$Z(x,y) = \int dW e^{(z-\bar{)}\mathrm{Tr}WW^\dagger} \sim \frac{1}{(z-\bar{z})^{n^2}} \sim \frac{1}{(x-\bar{y})^{n^2}}. \tag{145}$$

We thus find the correlator

$$\lim_{n\to 0} \frac{1}{n^2 N^2} \frac{d^2}{dxdy} \log Z(x,y) = -\frac{1}{N^2(x-y)^2}. \tag{146}$$

6 Spectral Form Factor of the Circular Unitary Ensemble

The Circular Unitary Ensemble (CUE) is the ensemble of random Unitary $N \times N$ matrices. It eigenvalues are on the complex unit circle, $\exp(i\theta_k)$ with joint probability distribution given by

$$\prod_{k<l}(e^{i\theta_k} - e^{i\theta_l}) \prod_{k<l}(e^{-i\theta_k} - e^{-i\theta_l}). \tag{147}$$

The Vandermonde determinants can be written as

$$\begin{aligned} &\prod_{k<l}(e^{i\theta_k} - e^{i\theta_l}) \det e^{i(k-1)\theta_l}\Big|_{1\le k,l\le N} \\ &\prod_{k<l}(e^{-i\theta_k} - e^{-i\theta_l}) \det e^{-i(k-1)\theta_l}\Big|_{1\le k,l\le N} \end{aligned} \tag{148}$$

Now we are ready to calculate the spectral form factor

$$\langle \mathrm{Tr}U^t \mathrm{Tr}(U^\dagger)^t \rangle. \tag{149}$$

To do that, we express the determinant as a sum over permutuations

$$\begin{aligned} &\det e^{i(k-1)\theta_l}\Big|_{1\le k,l\le N} \sum_\pi \mathrm{sign}(\pi) e^{i\pi_0\theta_1} \cdots e^{i\pi_{N=1}\theta_N}, \\ &\det e^{-i(k-1)\theta_l}\Big|_{1\le k,l\le N} \sum_\rho \mathrm{sign}(\rho) e^{-i\rho_0\theta_1} \cdots e^{-i\rho_{N=1}\theta_N}, \end{aligned} \tag{150}$$

and

$$\langle \mathrm{Tr}U^t \mathrm{Tr}(U^\dagger)^t \rangle \langle \sum_{k,l} e^{it(\theta_k - \theta_l)} \tag{151}$$

The normalization integral is only nonvanishing if $\pi = \rho$ and is thus given by $N!(2\pi)^N$. For $t = 0$ also on the case with $\pi = \rho$ gives a nonvanishing contribution, so that the result is given by N^2. For the form factor at $t \neq 0$ we have to distinguish the diagonal and off-diagonal terms. We have N diagonal terms, and in this case only $\pi = \rho$ gives a nonvanishing contribution which is equal to 1 for each of the terms (after dividing by the normalization. For $k \neq l$ all terms give the same contribution so that we can consider

$$N(N-1)\langle e^{it(\theta_1-\theta_2)}\rangle \tag{152}$$

We split the permutation in a permutation of $0, 1$ and the remaining $N-2$ numbers, for which we need to have $\pi|_{N-2} = \rho|_{N-2}$. The integral over $\theta_3, \cdots, \theta_N$ then gives $(2\pi)^{N-2}(N-2)!$. For the first two we can have either

$$\pi_0 = \rho_0, \cdots \pi_1 = \rho_1, \tag{153}$$

or

$$\pi_0 = \rho_1, \cdots \pi_1 = \rho_0. \tag{154}$$

In the second case we have an minus sign due to the signs of the permuations. So we have to calculate the expectation value

$$\langle e^{i\theta_1(t+\pi_0-\rho_0)} e^{i\theta_1(-t+\pi_1-\rho_1)} \tag{155}$$

11 Itzykson-Zuber integrals and the Duistermaat-Heckman theorem

In this chapter we discuss Harish-Chandra-Itzykson-Zuber integrals which, in the case of the $SU(N)$ groups are given by [9]

$$\int dU e^{i\mathrm{Tr}XU^{-1}YU} = c\frac{\det e^{ix_k y_l}}{\Delta(x)\Delta(y)}. \tag{156}$$

Here, the integral is over $SU(N)$ and X and Y are Hermitean matrices with eigenvalues x_i and y_i and c is a normalization constant. The Vandermonde determinants are given by

$$\Delta(x) = \prod_{k<l}(x_k - x_l). \tag{157}$$

The general case of this remarkable formula was first derived by Harish-Chandra [13]. Below we will derive this result in two different ways. First, via a diffusion equation, and, second, by means of a saddle-point approximation. Remarkably, the saddle-point result turns out to be exact. This is an example of the Duistermaat-Heckman [12] theorem which states under which conditions a saddle-point approximation becomes exact.

11.1 Derivation of the Itzykson-Zuber integral.

We consider the integral

$$\xi(A,t) = \frac{1}{(2\pi t)^{-n^2/2}} \int dB e^{-\frac{1}{2t}\mathrm{Tr}(A-B)^2} \eta(B), \tag{158}$$

with A and B Hermitean matrices, and $\eta(B)$ is an invariant function of B. By expressing the trace in components one can easily show that ξ satisfies the diffusion equation

$$\partial_t \xi = \frac{1}{2} \nabla_A^2 \xi, \tag{159}$$

with initial condition

$$\xi(A,0) = \eta(A). \tag{160}$$

The matrices A and B can be diagonalized by a unitary transformation

$$A = UXU^{-1}, \qquad B = VYV^{-1}. \tag{161}$$

The nonzero matrix elements of the diagonal matrices X and Y will be denoted by x_i and y_i, respectively. The Jacobian from the A variables to the U and X variables is given by

$$dA = \Delta^2(X) dX d\Omega_A. \tag{162}$$

where the angular degrees of freedom arec contained in Ω_A. In terms of these coordinates the Laplacian is given by

$$\nabla_A^2 = \frac{1}{\Delta^2(x)} \partial_{x_i} \Delta^2(x) \partial_{x_i} + \nabla_{\Omega_A}^2. \tag{163}$$

For an invariant function $\eta(B)$, the function $\xi(A,t)$ depends only on the eigenvalues of A and thus satisfies the equation

$$\partial_t \xi = \frac{1}{2} \frac{1}{\Delta^2(x)} \partial_i \Delta^2(x) \partial_i \xi \tag{164}$$

with initial condition

$$\xi(X,0) = \eta(X). \tag{165}$$

To solve this equation we make the substitution

$$F(X,t) = \Delta(x) \xi(X,t). \tag{166}$$

The second derivative of F is given by

$$\partial_i^2 F = \Delta \partial_i^2 \xi + 2 \partial_i \Delta \partial_i \xi + \xi \partial_i^2 \Delta. \tag{167}$$

The r.h.s. of this equation can be rewritten as

$$\Delta \frac{1}{\Delta^2} \partial_i (\Delta^2 \partial_i \xi). \tag{168}$$

This results in the diffusion equation

$$\partial_t F = \frac{1}{2}\partial_i^2 F. \tag{169}$$

The solution of this diffusion equation is given by

$$F(X,t) = c\frac{1}{t^{n/2}}\int dY \Delta(y) e^{-\frac{1}{2t}\sum_i (x_i - y_i)^2}\eta(Y), \tag{170}$$

with c an arbitrary constant. An invariant function is also a symmetric function of the eigenvalues. The exponent in this equation can be factorized in symmetric functions of the integration variables and $\exp(\sum_i x_i y_i/t)$. Only the antisymmetrized part of the latter term contributes to the integral. The result for $F(X,t)$ is be valid for any invariant function $\eta(Y)$ of the eigenvalues and the coefficients of $\eta(B)$ in Eq. (158) and the ones that follows from $F(X,t)$ should be the same after antisymmetrization the exponential function. After multiplying out the squares in the exponent of both sides of the equation we obtain

$$\int dU e^{-\frac{1}{2t}\mathrm{Tr}UXU^{-1}VYV^{-1}} = ct^{(n^2-n)/2}\frac{\det e^{-\frac{1}{2t}x_k y_l}}{\Delta(y)\Delta(x)}, \tag{171}$$

which is the final result for the Itzykson-Zuber integral.

11.2 Semiclassical calculation of the Itzykson-Zuber integral.

In this section we calculate the Itzykson-Zuber integral by means of a saddle-point approximation. The saddle-point equations are given by

$$d\mathrm{Tr}XU^{-1}YU = 0. \tag{172}$$

Using the product rule and $dU^{-1} = -U^{-1}dUU^{-1}$ we obtain

$$\mathrm{Tr}XU^{-1}dUU^{-1}YU = \mathrm{Tr}XU^{-1}YdU. \tag{173}$$

By equating the coefficient of dU on both sides of the equation we obtain

$$XU^{-1}YU = U^{-1}YUX. \tag{174}$$

The solution of this equation is a permutation matrix, i.e.

$$UXU^{-1} = X_\pi, \tag{175}$$

where X_π is the diagonal matrix with diagonal elements $x_{\pi(i)}$. Of course we have

$$U^{-1}YU = Y_{\pi^{-1}}, \tag{176}$$

which immediately proves our assertion that permutation matrices are solutions of our saddle-point equations. From the definition of the integral it is clear that $U \in U(N)/U^N(1)$, and saddle-points that only differ by a diagonal unitary matrix have not to be taken into account.

Let us next expand around the saddle-point,

$$U = \bar{U}e^{i\phi} = \bar{U}(1 + i\phi - \frac{1}{2}\phi^2 + \cdots), \tag{177}$$

where $\phi^\dagger = \phi$. To second order in ϕ we obtain

$$\begin{aligned} \mathrm{Tr}XU^{-1}YU &= \mathrm{Tr}X\bar{U}^{-1}Y\bar{U} \\ &+ -\frac{1}{2}\mathrm{Tr}X\phi^2\bar{U}^{-1}Y\bar{U} - \frac{1}{2}\mathrm{Tr}X\bar{U}^{-1}Y\bar{U}\phi^2 + \mathrm{Tr}X\phi\bar{U}^{-1}Y\bar{U}\phi. \end{aligned} \tag{178}$$

The terms linear in ϕ vanish because of the saddle-point equations. Notice that $\bar{U}^{-1}Y\bar{U}$ and X are both diagonal and commute. In terms of components the terms of second order in ϕ can be written as

$$-\sum_{i<k}\phi_{ik}\phi^*_{ik}(Y_{\pi(i)} - Y_{\pi(k)})(X_i - X_k), \tag{179}$$

where we have used the Hermiticity of ϕ. Summing over all saddle-points we obtain

$$\sum_\pi \frac{e^{\sum_i X_i Y_{\pi(i)}}}{\prod_{i<k}(X_i - X_k)(Y_{\pi(i)} - Y_{\pi(k)})}. \tag{180}$$

If we use that

$$\prod_{i<k}(Y_{\pi(i)} - Y_{\pi(k)}) = (-1)^\pi \prod_{i<k}(Y_i - Y_k), \tag{181}$$

we obtain our desired result

$$\int dU e^{i\mathrm{Tr}XU^{-1}YU} = c\frac{\det e^{x_k y_l}}{\Delta(X)\ \Delta(Y)}. \tag{182}$$

It was first pointed out by Mike Stone [11], that the exact result coincides with the result obtained by a saddle-point approximation.

11.3 Invariant Measure for $U(2)/U(1)\times U(1)$

A $U(2)/U(1)\times U(1)$ matrix can be parameterized as

$$U = \begin{pmatrix} x\cos\theta & -\sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix}. \tag{183}$$

To find the invariant measure, we have to calculate $U^{-1}dU$. It is given by

$$U^{-1}dU = \begin{pmatrix} i\sin^2\theta d\phi & (id\phi\cos\theta\sin\theta - d\theta)e^{-i\phi} \\ (id\phi\cos\theta\sin\theta + d\theta)e^{-i\phi} & -i\sin^2\theta d\phi \end{pmatrix} \tag{184}$$

We have to calculate the Jacobian from the $\theta\phi$-variables to the $(U^{-1}dU)_{12}$ and $(U^{-1}dU)_{21}$ variables. It is equal to

$$J = 2|\cos\theta\sin\theta|. \tag{185}$$

11.4 The Itzykson-Zuber integral for $SU(2)$

In the parametrization (183, the exponent in the Itzykson-Zuber integral is given by

$$\mathrm{Tr}UXU^{-1}Y = x_1y_2 + x_2y_1 + (x_1 - x_2)(y_1 - y_2)\cos^2\theta. \tag{186}$$

This results in the Itzukson-Zuber integral

$$I = 2\int_0^\pi d\theta \int_0^{2\pi} \frac{d\phi}{2\pi} |\cos\theta \sin\theta| e^{x_1y_2+x_2y_1+(x_1-x_2)(y_1-y_2)\cos^2\theta}. \tag{187}$$

The integral over ϕ is trivial. The integrand has critical points at $\theta = 0$, $\theta = \pi/2$ and $\theta = \pi$. The integrand is a total derivative on each of these intervals separately, but cannot be written as a total derivative globally. So we have

$$\begin{aligned} I &= \frac{1}{2}\int_0^{\pi/2} d\theta \frac{d}{d\theta} \frac{e^{x_1y_2+x_2y_1+(x_1-x_2)(y_1-y_2)\cos^2\theta}}{(x_1-x_2)(y_1-y_2)} \\ &\quad -\frac{1}{2}\int_{\pi/2} 2^{\pi/} d\theta \frac{d}{d\theta} \frac{e^{x_1y_2+x_2y_1+(x_1-x_2)(y_1-y_2)\cos^2\theta}}{(x_1-x_2)(y_1-y_2)}. \end{aligned} \tag{188}$$

The result of the integral due to the endpoints and is thus given by

$$\frac{2(e^{x_1y_1+x_2y_2} - e^{x_1y_2+x_2y_1})}{(x_1-x_2)(y_1-y_2)}, \tag{189}$$

which is the correct result for the for the Itzykson-Zuber integral. We observe that the integral is localized at the critical points which coincide with the saddle-points of the integrand.

11.5 Connection with the Duistermaat-Heckman Theorem.

It is no accident that the saddle-point approximation for the Itzykson-Zuber integral gave the correct result. The general conditions for the semi-classical exactness of phase-space integrals were outlined in a theorem by Duistermaat and Heckman. The simplest example of this theorem is given by the integral

$$\int_0^\pi d\theta \sin\theta e^{a\cos(\theta)} = -\frac{2\pi}{a}(e^a - e^{-a}). \tag{190}$$

The exact answer for this integral follows immediately by observing that the integrand is a total derivative. However, the result of this integral is entirely localized on the saddle-points and is therefore semiclassical exact. Because the integrand vanishes on the saddle-points, the saddle-point integration has to be performed with care and is given by

$$2\pi\int_0^\infty \theta e^{a-a\theta^2/2} + \int_{-\infty}^\pi (\pi-\theta)e^{-a+a(\pi-\theta)^2/2} = \frac{2\pi}{a}(e^a - e^{-a}). \tag{191}$$

Since the integrand is localized at the critical points it suffices the expand the integrand up to second order at these points.

Both this example and the Itzykson-Zuber integral for $SU(2)$ are of the general type

$$I = \int d\theta |f'(\theta)| e^{f(\theta)}. \tag{192}$$

Locally the integrand is a total derivative and the contributions are localized on the critical points, i.e. on the points where the Jacobian vanishes. The result of the integral is given by

$$I = -2e^{f(\theta_0)} + 2e^{f(\theta_1)} - 2e^{f(\theta_2)} + 2e^{f(\theta_3)} - \cdots. \tag{193}$$

However the points $f'(\theta_i) = 0$ are also saddle-points. Expanding to second order around the saddle-points we obtain

$$I = \sum_i \int d\theta \alpha_i |\theta - \theta_i| e^{f(\theta_i) - \frac{1}{2}(\theta-\theta_i)^2 f''(\theta_i)}, \tag{194}$$

where $f(\theta) = \alpha_i(\theta - \theta_i)$ in the neigborhood of a critical point. The saddle-point result for the integral is thus given by

$$-2\frac{f'(\theta_0)}{f''(\theta_0)} + 2\frac{f'(\theta_1)}{f''(\theta_1)} = 2\frac{f'(\theta_2)}{f''(\theta_2)} + 2\frac{f'(\theta_3)}{f''(\theta_3)} - \cdots. \tag{195}$$

We observe that the saddle-point integration is exact if $\alpha = f''(\theta_i)$. One can easily verify that this condition is satisfied for the examples considered above.

In general we have an integrand that can locally be written as a total derivative. However is we exclude the critical points in the integration domain the integrand can be written as a total derivative globally.

$$I = \sum_k \int_{C_\epsilon^k} d\omega = \sum_k \int_{\partial C_\epsilon^k} \omega, \tag{196}$$

where the C_ϵ^k are spheres of radius ϵ around the critical points. The integrand is therefore localized at the critical points. At these points the saddle-point approximation becomes exact. In coordinates where the mixed derivatives vanish the condition for the exactness of the saddle-point approximation is that the ratio of the product of the α_i and the product of the second derivatives is equal to unity.

As an example let us again consider

$$\int sin\theta d\theta d\phi e^{a\cos\theta} = -\frac{1}{a}\int d(d\phi e^{a\cos\theta}) = -\frac{1}{a}\int_{\partial C(0,\epsilon)} d\phi e^{a\cos\theta} - \frac{1}{a}\int_{\partial C(\pi,\epsilon)} d\phi e^{a\cos\theta}. \tag{197}$$

Taking into account that the orientation of C_ϵ is opposite on both sided of the sphere, we again find the same final answer. The reason that an infinitesimal integration interval gives a finite ansewere is that ϕ varies by a finite amount (2π) around this circle. This happens only at the North pole and the South pole of the sphere.

For the case of $SU(2)$ we also have a critical point at $\theta = \pi/2$. The reason is that at this point the unitary martrices become independent of α.

12 Eigenvectors

For the invariand random matrix theories, the eigenvectors are distributed according to the Haar measure. For the Gaussian Orthogonal ensemble, this means that the joint distribution of the components is given by

$$\delta(\sum_\alpha c_\alpha^2 - 1). \tag{198}$$

The distribution of a single component is obtained by integrating over the remaining ones. This can be done by changing to $2N$-dimensional polar coordinates,

$$\begin{aligned} x_1 &= \cos\theta, \\ x_2 &= \sin\theta\cos\phi_1, \\ x_3 &= \sin\theta\sin\phi_1, \\ x_4 &= \sin\theta\sin\phi_1\cos\phi_2, \\ x_5 &= \sin\theta\sin\phi_1\sin\phi_2, \\ &\cdots \end{aligned} \tag{199}$$

The measure is given by

$$d\Omega = \sin^{2N-2}\theta\sin^{2N-3}\phi_1\cdots\sin\phi_{2N-3}. \tag{200}$$

so that the probability of the first component is given by

$$P(x)dx = \delta(x-\cos\theta)\sin^{2N-2}\theta dx = (1-x^2)^N dx \approx e^{-x^2/N}. \tag{201}$$

This distribution is know as the Porter-Thomas distribution.

Another measure of eigenvector components is the so called strength distribution or the local density of states (LDOS). If eigenvector are expanded in a basis as

$$\psi_k = \sum_\alpha c_{k\alpha}\phi_\alpha, \tag{202}$$

it is defined as

$$\rho_\alpha(E) = \sum_k |c_{k\alpha}|^2\delta(E-E_k). \tag{203}$$

For the invariant random matrix ensembles, the eigenvalues and eigenvectors are statistically independent so that the average LDOS is equal to the average density of states.

12.1 Survival Probability

The time evolution of a quantum state initially in state $\psi(0)$ is given by

$$\phi_\alpha(t) = e^{-iHt}\phi_\alpha(0) = \sum c_{k\alpha}e^{-iE_k}\psi_k. \tag{204}$$

The survival probability or return probability is defined as

$$W_\alpha = |\langle\psi(t)|\phi_\alpha(0)\rangle|^2 = \left|\sum |c_{k\alpha}|^2 e^{-iE_k}\psi_k\right|^2. \tag{205}$$

It can be expressed in the local density of states as

$$W_\alpha = \left| \int e^{-iEt} \rho_\alpha(E) \right|^2 . \tag{206}$$

If we average W_α over all basis states, we get the spectral form factor

$$\frac{1}{N} \sum W_\alpha = \frac{1}{N} \left| \sum_k e^{-iE_k t} \psi_k \right|^2 . \tag{207}$$

12.2 Participation Ratio

The participation ration is defined as

$$\sum_\alpha |c_{k\alpha}|^4 / \sum_\alpha |c_{k\alpha}|^2. \tag{208}$$

We normally assume that the states are normalized to 1, so the the denominator can be ignored. A more useful quantity is the inverse participation ratio (IPR) which is define as

$$IPR = \frac{1}{\sum_\alpha |c_{k\alpha}|^4}. \tag{209}$$

If we have a state of length N with $N - p$ components equal to zero, and p components equal to $1/\sqrt{p}$ it is given by

$$IPR = p. \tag{210}$$

Another observable to charactetize wave functions is the multi-fractality index τ_q. It is defined as

$$P_q = \frac{\sum_\alpha |c_\alpha|^{2q}}{P \sum_\alpha |c_\alpha|^2} = N^{-\tau_q}. \tag{211}$$

For the Porter-Thomas distribution we obtain

$$\tau_q = q - 1. \tag{212}$$

If we have a lattice of dimension d, then $N = D^d$, and the scaling dimension is given by $d(q-1)$. The anaomalous multifractality index is given by

$$\tau_q = d(q-1) + \Delta_q. \tag{213}$$

13 Quantum Chaos and Random Matrix Theory

The first connection was made by means of the study of the Sinai billiard which is classically chaotic. It would found by Bohigas, Giannoni and Schmidt that its are correlated by Random Matrix Theory. This led to what is now know as the *Bohigas-Giannoni-Schmidt conjecture*:

> *If a system is classically chaotc its eigenvalues are correlated according to Random Matrix Theory* .

13.1 Semiclassical Argument

The semiclassical limit is the limit $\hbar \to 0$. In this limit, the level density becomes very large or conversely, the level spacing approaches zero. In the limit, the spectral density can be decomposed as

$$\rho(E) = \bar{\rho}(E) + \rho_{\rm fluc} \tag{214}$$

where $\bar{\rho}(E)$ is obtained by locally smoothing of the spectral density which is well defined in the semiclassical limit. The fluctuating part of the spectral density in the $\hbar \to 0$ limit is given by [**?**]

$$\rho_{\rm fluc} = \frac{1}{\pi\hbar}{\rm Re}\sum_j A_j(E)e^{iS_j(E)/\hbar} \tag{215}$$

The sum is over all periodic trajectories includng multiple traversals. The spectral form factor is given by

$$K(\tau) = \int \frac{d\epsilon}{\bar{\rho}} e^{2\pi\tau\epsilon\bar{\rho}} \frac{1}{2\pi^2\bar{h}^2}\sum_{jj'} A_j A_{j'}^* e^{i(S_{j'}(E+\epsilon/2)-S_j(E-\epsilon/2)/\hbar} \tag{216}$$

We consider the microscopic limit where ϵ scales as the average lebel spacing. Then we can Taylor expand the action

$$S_{j'}(E+\epsilon/2) - S_j(E-\epsilon/2) \approx S_{j'}(E) - S_j(E) + \epsilon\frac{dS_j}{dE} = \epsilon T_j. \tag{217}$$

In the semiclassical limit, $\hbar \to 0$, the terms $\exp(i(S_j(E) - S_{j'}(E))/\hbar)$ are strongly oscillating for $j \neq j'$. So the leading contributions come from the terms with $j = j'$. This is known as the diagonal approximation. We thus find

$$\begin{aligned} K(\tau) &= \int \frac{d\epsilon}{\bar{\rho}} e^{2\pi\tau\epsilon\bar{\rho}} \frac{1}{2\pi^2\bar{h}^2}\sum_j A_j^2 e^{i\epsilon T_j/\hbar} \\ &= \frac{1}{2\pi\bar{\rho}\hbar}\frac{1}{\pi}\sum_j \delta(2\pi\hbar\bar{\rho}\tau - T_j)A_j^2. \end{aligned} \tag{218}$$

Next we consider the quantity

$$\sum_j \delta(T - T_j)A_j^2, \tag{219}$$

which is known as the Hannay-Ozorio de Almeida sum rule, and was first applied to the calculation of the spectral form factor by Michael Berry [**?**]. It is conceptually simpler to consider the average

$$\frac{1}{T}\sum_j \delta(T - T_j)A_j^2 f(p,q), \tag{220}$$

which is the average of $f(p,q)$ along the trjectory in phase space. To have a well-defined average, we have to consider a group of trajectories with period no more than ΔT away from T. For a

chaotic system a classical trajectory covers all of phase space, and we know assume that the trajectory covers the phase space also uniformely in time so that the sum (220) gives the phase-space average of the function $f(p,q)$ on the energy shell. For $f=1$ we have

$$\frac{1}{T}\sum_j \delta(T-T_j)A_j^2 = 1. \tag{221}$$

which is the Hannay-Ozorio de Almeida sum rule. Using this result, we find

$$K(\tau) = \frac{\tau}{\pi}. \tag{222}$$

13.2 The Lieb-Robinson Bound

We consider a local bounded Hamiltonian H and derive an inequality for

$$\parallel [e^{iHt}A(x)e^{-iHt}, B(0)] \parallel . \tag{223}$$

From the Baker-Campbell-Hausdorff formula we obtain

$$e^{iHt}A(x)e^{-iHt} = A(x) + it[H,A(x)] + \frac{(it)^2}{2}[H,[H,A(x)]] + \cdots. \tag{224}$$

If the Hamiltonian has a reach of R the first term that contributes is for $k_{\min} \approx x/R$. This results in the bound

$$\begin{aligned} & 2 \parallel A(x) \parallel\parallel B(0) \parallel N_{\text{paths}} \sum_{k=k_{\min}}^{\infty} \frac{|(it)^k|}{k!} e^{-\mu k} \\ \leq\ & 2N_{\text{paths}} \parallel A(x) \parallel\parallel B(0) \parallel e^{-\mu x/R} \sum_{k=0}^{\infty} \frac{|(t)^k|}{k!} \\ \leq\ & 2N_{\text{paths}} \parallel A(x) \parallel\parallel B(0) \parallel e^{(v_{LB}t-x)\mu/R}, \end{aligned} \tag{225}$$

where $\exp(-\mu)$ is a bound for the hopping matrix elements of the Hamiltonian and v_{LB} is known as the Lieb-Robinson velocity or the butterfly velocity. Information cannot propagate outside the Lieb-Robinson cone. This indeed has been observed numerically as well as in some experiments.

The Lieb-Robinson bound is only valid for small times. We also have the upper limit

$$2 \parallel A(x) \parallel\parallel B(0) \parallel . \tag{226}$$

13.3 Out of Time Order Correlators (OTOC)

What we have seen in the previous subsection is an example of an out of time order correlator. For conjugate variables we have

$$[p(t),x(0)]^2 \sim \{p(t),x(0)\}_0^2 = \left|\frac{\delta p(t)}{\delta p(0)}\right|^2. \tag{227}$$

For a chaotic system the trajectories diverge exponentially so that

$$\left|\frac{\delta p(t)}{\delta p(0)}\right| \sim e^{\lambda t} \tag{228}$$

where λ is known as a Lyapunov exponent. We thus find that out of time order correlators increase exponentially but the increase is bounded by the Lieb-Robinson bound.

14 Thermalization

The traditional way of defining thermalization is to emmerse a system in a much large heat bath, and it is thermalized when the heat exchange with the heat bath has saturated. For a closed system we cannot apply this argument.

The state of a system can be characterized by the density matrix

$$\rho = \sum a_k |k\rangle\langle k|, \tag{229}$$

which is normalized as $\mathrm{Tr}\rho = 1$. The time evolution of the density matrix is given by

$$\rho(t) = e^{-iHt}\rho(0)e^{iHt}. \tag{230}$$

We consider a closed system with many more degrees of freedom in B, the complement of A, than in A. To define the thermodynmic limit we have to take the number of degrees of freedom of B to infinity. The equilibrium density matrix is given by

$$\rho_{\mathrm{eq}}(T) = \frac{1}{Z}e^{-\beta H} = \frac{1}{Z}\sum_k e^{-\beta E_k}|k\rangle\langle k|. \tag{231}$$

where $\beta = 1/k_B T$ and Z is the partition function at temperature Z. The eigenstates of the Hamiltonian are denoted by $|k\rangle$ with eigenvalues E_k. A system has thermalized if for all subsystems A the long time, large system limit is given by

$$\rho_A(T) = \rho_{\mathrm{A,eq}}(T) \tag{232}$$

with the density matrix of the subsystem given by $\rho_A = \mathrm{Tr}_B\rho$.

14.1 Eigenstate Thermalization

If a single eigenstate thermalizes the temperature is given by

$$E_k = \langle H\rangle_{T_k}. \tag{233}$$

We speak about eigenstate thermalization is for all subsystems A,

$$\rho_{k,A}\rho_{A,eq}(T_k) \tag{234}$$

with

$$\rho_{k,A} = \mathrm{Tr}_B|k\rangle\langle k|. \tag{235}$$

Eigenstate thermalization can also be formulated in terms of expectation values of operators [14],

$$O_{mn} = O(\bar{E})\delta_{mn} + e^{-S(\bar{E})/2}f_O(\bar{E},\omega)R_{mn} \tag{236}$$

with

$$\bar{E} = \frac{1}{2}(E_m + E_n), \qquad \omega = E_n - E_m, \tag{237}$$

and $S\bar{E}$) the entropy of the system. What is important that $O(\bar{E})$, $S(\bar{E})$ and $f_O(\bar{E},\omega)$ are smooth functions of $\bar{E}$ and $\bar{\omega}$. R_{mn} is a random variable wih zero mean and unit variance.

The thermal average of an operator in A is given by

$$\bar{O} = \mathrm{Tr}_A\rho_{\mathrm{eq}}(T)O = \mathrm{Tr}_A\rho_A O\mathrm{Tr}_A\mathrm{Tr}B\rho O = \langle n|O|n\rangle, \tag{238}$$

which agrees with the definition of previous section.

15 The Sachdev-Ye-Kitaev Model

This model, that is better know as the SYK model is defined by the Hamiltonian of N interacting Majorana fermions,

$$H = \frac{1}{q!} \sum_{1 \le \alpha\beta\gamma\delta \le N} J_{\alpha\beta\gamma\delta} \chi_\alpha \chi_\beta \chi_\gamma \chi_\delta, \tag{239}$$

where the $J_{\alpha\beta\gamma\delta}$ are Gaussian distributed with variance given by

$$\langle J^2_{\alpha\beta\gamma\delta} \rangle = \frac{J^2 (q-1)!}{N^{q-1}}, \tag{240}$$

and $q = 4$. The Majorana fermions satisfy the anti-commutation relations

$$\{\chi_\alpha, \chi_\beta\} = \delta_{\alpha\beta}. \tag{241}$$

These are the same anti-commutation relations as the Euclidean γ matrices, and therfore, the χ_α can be represented a γ matrices in N dimensions. In general the interaction can be a q-body ineraction, but in the above Hamiltonian (239) we have $q = 4$. Generally, we will take N and q even, but this can be relaxed. For odd q the Hamiltonian is the supercharge of a super-symmetric Hamiltonian.

For even N we have a well defined γ_5 that anti-commutes with all γ matrices so that

$$[H, \gamma_5] = 0. \tag{242}$$

This implies that in a chiral basis, the Hamiltonian splits into two blocks.

15.1 Gamma Matrices

Gamma matrices in N dimensions can be constructed as tensor products of $N/2$ Pauli matrices, and are $2^{N/2} \times 2^{N/2}$ dimensional matrices. In total there are $4^{N/2}$ different γ matrices so that each they span the space of $2^{N/2} \times 2^{N/2}$ matrics. An example of an explicit representation of γ matrices is given by

$$\gamma^{(2)}_1 = \sigma_1, \qquad \gamma^{(2)}_2 = \sigma_2, \qquad \gamma^{(2)}_3 = \sigma_3. \tag{243}$$

and using the recursion relation

$$\begin{aligned} \gamma^{(d+2)}_k &= \sigma_1 \otimes \gamma^d_k, \qquad \text{for} \quad k = 1, \cdots, d+1, \\ \gamma^{d+2}_{d+2} &= \sigma_2 \otimes 1_{2^{d/2}}. \end{aligned} \tag{244}$$

We can construct two anti-unitary symmetry operators (Note that the gamma matrices in C_1 are putrely imaginary while the gamma matrices in C_2 are purely real.)

$$\begin{aligned} C_1 &= \gamma_1 \gamma_4 \gamma_6 \cdots \gamma_N K, \\ C_2 &= \gamma_2 \gamma_3 \gamma_5 \cdots \gamma_{N-1} K, \end{aligned} \tag{245}$$

where K is the complex conjugation operator (we really should have interchanged the labels of γ_1 and γ_2 so that C_1 would have been the product of the odd gamma matrices and C_2 the product of the even gamma matrices). They satisfy the symmetry relations

$$C_1 K \gamma_\mu + (-1)^{N/2} \gamma_\mu C_1 K = 0, \qquad C_2 K \gamma_\mu - (-1)^{N/2} \gamma_\mu C_1 K = 0. \tag{246}$$

Since the Hamiltonian is sum of products of four gamma matrices, we have

$$[C_1 K, H] = 0, \qquad [C_2 K, H] = 0. \tag{247}$$

We also have that

$$[C_1 K, C_2 K] = 0. \tag{248}$$

Since $C_1 K C_2 K \sim \gamma_5$, these two symmetries are not independent, and we can choose one of them as anti-unitary symmetry. We will choose $C_2 K$ as anti-unitary symmetry operator. If $N/2$ is even we have that $[\gamma_5, C_2 K] = 0$, so that γ_5 and $C_2 K$ can be diagonalized simultaneously. Depending on the sign of $(C_2 K)^2 = \pm 1$, this gives the GOE or the GSE. In four dimensions we have $(C_2 K)^2 = -1$ while in 8 dimensions $(C_2 K)^2 = 1$. In $4k + 2$ dimensions, the charge conjugation operator is the product of $2k + 1$ γ matrices and anti-commute with γ_5, and cannot be diagonalized simultaneously. In a chiral basis, the matrix elements of the Hamiltonian are complex and we expect GUE level statistics.

Since the charge conjugation operator anti-commutes with γ_5, it has the form

$$C_2 K = \begin{pmatrix} 0 & c_2 K \\ c_2^* K & 0 \end{pmatrix} \tag{249}$$

with $c_2^* c_2 = -1$. If

$$H \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{250}$$

then the anti-unitary symmetries (247) result in the relation

$$B^* = -c_i^* A c_i, \qquad i = 1, 2. \tag{251}$$

with $c_2^* c_2 = -1$. If

$$H \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \tag{252}$$

then the anti-unitary symmetries (247) result in the relation

$$B^* = -c_2^* A c_2, \tag{253}$$

From the eigenvalue equation $\det(B - \lambda) = 0$, we then see immediately that A and B have the same eigenvalues.

15.2 Hilbert Space

We can define creation and annihilation operators by

$$c_k = \frac{1}{\sqrt{2}}(\gamma_{2k} i + \gamma_{2k+1}), qquad c_k^\dagger = \frac{1}{\sqrt{2}}(\gamma_{2k} i - \gamma_{2k+1}). \tag{254}$$

Then

$$\{c_k, c_l^\dagger\} = \delta_{kl}. \tag{255}$$

So the $c_k^\dagger$ are the creation operators of complex fermions. The Hilbert space is the usual Fock space with state k are empty or occupied. So in total we have $2^{N/2}$ states. Note that the SYK Hamiltonian does not converse particle number, as can be seen by substituting the inverse of (254) in the SYK Hamiltonian.

15.3 The $q = 2$ Hamiltian

The SYK Hamiltonian for q = 2 model is given by

$$H = i \sum_{i<j} J_{ij} \gamma_i \gamma_j. \tag{256}$$

Since J_{ij} is an anti-symmetric matrix, it can be brought into a standard form with only $J'_{k,k\pm1} \neq 0$. Using the unitary transformation that diagonalizes J to define new γ matrices, we can write the Hamiltonian as

$$H = i \sum_{k=1}^{N/2} x_k \gamma_{2k-1} \gamma_{2k} \tag{257}$$

with x_k the eigenvalues of J. All terms in the new Hamiltonian commute with each other and can be diagonalize simultaneously. The eigenvalues of $\gamma_{2k-1}\gamma_{2k}$ are ± 1. The eigenvalues of the Hamiltonian are thus given by

$$\sum_{k=1}^{N/2} s_k x_k \tag{258}$$

with $s_k = \pm 1$. By rewriting the Hamiltonian in terms of complex fermions

$$H = \sum_{k=1}^{N/2} x_k (2 c_k c_k^\dagger - 1). \tag{259}$$

we see that the ground state is the state with all negative energy states filled.

15.4 Thermodynamics for $q = 2$

For $q = 2$ the SYK Hamiltonian is non-interacting with single particle energies given by $\pm x_k$. The results in the partition function

$$\begin{aligned} Z = \sum_{n_k=0,1} e^{-\beta \sum_{k=1} N x_k (2n_k - 1} &= \prod_{k=1}^{N} (e^{\beta x_k} + e^{-\beta x_k}) \\ &= \prod_{k=1}^{N} e^{\beta x_k} \prod_{k=1}^{N} (1 + e^{-2\beta x_k}). \end{aligned} \tag{260}$$

So

$$\log Z = \beta E_0 + \int_{-1}^{1} \rho(E) dE (1 + e^{-2\beta E}) \tag{261}$$

with E_0 the sum of the single particle energies and the single particle level density

$$\rho(E) = \frac{N}{\pi} \sqrt{1 - E^2}. \tag{262}$$

We thus have that

$$\log Z = \beta E_0 + \frac{N}{\pi} \int_0^{\pi} d\theta \cos^2 \theta \log(1 + e^{-2\beta \sin \theta}). \tag{263}$$

The low temperature limit is obtained by expanding the logarithm

$$\log Z = \beta E_0 + N \frac{\pi}{12\beta}. \tag{264}$$

So the zero temperature entropy, which would be the constant term in $\log Z$ vanishes. The hightemperature limit of the partition function is given by

$$\log Z = \beta E_0 + \frac{N}{2} \log 2 + \cdots, \tag{265}$$

which is the logarithm of the total number of states.

15.5 Calculation of the Spectral Density

The most straigtforward way to calculate the spectral density is to use the moments to obtain the Fourier transform of the spectral density,

$$\rho(t) = \int dE \rho(E) e^{iEt} = \sum \frac{(Et)^{2k} (-1)^k}{k!} M_{2k}, \tag{266}$$

where we have assumed that $\rho(E)$ is an even function of E. In the limit that $N \gg q$, the indices of the terms contributing to the Hamiltonian are almost always different, and we can assume that the Γ_α commute. Summing over all Wick contractions we obtain

$$M_{2p} = (2p - 1)!! M_2^p. \tag{267}$$

These are the moments of a Gaussian distribution.

15.6 Path Integral Formulation of the SYK Model

For Dirac fermions we know very well how to write down the path integral of fermion fields. Just replace the fermion operators by complex Grassmann variables. For Majorano fermions we replace the real fermion operators, which can be written as $\chi = c + c^\dagger$, by real Grassmann variables. The kinetic term of the Lagrangian is given by

$$L_0 = \int d\tau \chi(\tau) \frac{d}{d\tau} \chi(\tau). \tag{268}$$

We thus obtain the path integral

$$Z = \int D\chi(\tau) e^{-\int d\tau \chi \frac{d}{d\tau}\chi - \sum_{\alpha\beta\gamma\delta} J_{\alpha\beta\gamma\delta}\chi_\alpha\chi_\beta\chi_\gamma\chi_\delta}. \tag{269}$$

The free energy can be obtained using the replica trick

$$\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n}. \tag{270}$$

To calculate Z^n we given the fields an extra index, the replica index. However, since thermodynamic properties do no depend on the number of replicas, we will do the calculation for one replica.

The integral over the Gaussian random variables can be done by completing square. Denoting the variance by σ^2 and using

$$\int dJ e^{-J^2/2\sigma^2 + JA} \sim e^{\frac{1}{2}\sigma^2 A^2} \tag{271}$$

we obtain

$$Z = \int D\chi(\tau) e^{-\int d\tau \chi \frac{d}{d\tau}\chi + \frac{1}{2}\sigma^2 \sum_{\alpha\beta\gamma\delta}\int d\tau d\tau' \chi_\alpha(\tau)\chi_\beta(\tau)\chi_\gamma(\tau)\chi_\delta(\tau)\chi_\alpha(\tau')\chi_\beta(\tau')\chi_\gamma(\tau')\chi_\delta(\tau')} \tag{272}$$

Next we introduce new variables by inserting a δ function

$$\delta(G + \frac{1}{N}\sum_\alpha \chi(\tau)\chi(tau')) = \int D\Sigma(\tau) e^{\frac{1}{2}\int d\tau d\tau' \Sigma(\tau,\tau')(G(\tau,\tau') + \frac{1}{N}\sum_\alpha \chi(\tau)\chi(\tau'))}. \tag{273}$$

The integral over $\Sigma(\tau)$ has to be over the imaginary axis, but we continue it to real axis. We thus find the partition function (we wrote it down for arbitrary q)

$$Z = \int D\chi(\tau) e^{-\int_0^\beta d\tau \chi \frac{d}{d\tau}\chi + \frac{\sigma^2}{q!}\int_0^\beta d\tau d\tau' G(\tau,\tau')q + \frac{1}{2}\int d\tau d\tau' \Sigma(\tau,\tau')(G(\tau',\tau) + \frac{1}{N}\sum_\alpha \chi(\tau)\chi(tau')} \tag{274}$$

Now the the integral of $\chi(\tau)$ can be done. It gives a Pfaffian. The action is thus given by

$$S = -\int\int d\tau d\tau' \left[\frac{N}{2}\mathrm{Tr}\log(\delta(\tau,\tau') + \Sigma(\tau,\tau')) + \frac{N^4\sigma^2}{q!}G^q(\tau,\tau') + N\Sigma(\tau,\tau')G(\tau',\tau)\right]. \tag{275}$$

Because of our choice of the variance, N only appears as a prefactor. For large N we can evaluate the integral by a saddle-point approximation.

$$\begin{aligned} \int d\tau''(\delta(\tau,\tau'')\frac{d}{d\tau} + \Sigma(\tau,\tau''))G(\tau'',\tau') &= -\delta(\tau,\tau', \\ \Sigma(\tau,\tau') &= J^2 G^{q-1}(\tau,\tau') \end{aligned} \tag{276}$$

The second equation is valid point by point. Because the source term is translationally invariant, the solutions of the Dyson-Schwinger equations are also translationally invariant. It is simplest to solve these equation by Fourier transforming the first equation

$$\begin{aligned} \Sigma(t-s) &= \frac{1}{\sqrt{2\pi}}\int dt e^{iE(t-s)}\Sigma(E), \\ G(s-t) &= \frac{1}{\sqrt{2\pi}}\int dE' e^{iE(s-t')}G(E'). \end{aligned} \tag{277}$$

After integration over s and then integrating $\delta(E-E')$, the first equation becomes

$$\int dE\Sigma(E)G(E)e^{iE(t-t')} = -\frac{1}{2\pi}\int dE e^{iE(t-t')}. \tag{278}$$

We thus find that

$$\Sigma(E)G(E) = -\frac{1}{2\pi}. \tag{279}$$

We make the Ansatz

$$G(E) = -ib\,\mathrm{sign}(E)/\sqrt{|E|}^{\alpha}. \tag{280}$$

Then

$$\Sigma(E) = -i\frac{1}{2\pi b}\mathrm{sign}(E)\sqrt{|E|}^{-\alpha} \tag{281}$$

The Fourier transform of $\mathrm{sign}(E)\sqrt{|E|}^{\alpha}$ is given by

$$\begin{aligned}
\frac{1}{\sqrt{\pi}}\int dE e^{-itE}\mathrm{sign}(E)\sqrt{|E|}^{\alpha} &= \frac{-i}{\sqrt{\pi}}\int dE\,\sin(tE)\mathrm{sign}(E)\sqrt{|E|}^{\alpha} \\
&= \frac{-2i}{\sqrt{\pi}}\int_0^{\infty} dE\,\sin(Et)\mathrm{sign}(t)E^{\alpha/2} \\
&= \frac{-2i}{\sqrt{\pi}}t^{-\frac{\alpha}{2}-1}\mathrm{sign}(t)\int_0^{\infty} dE\,\sin(E)E^{\alpha/2}E^{\alpha/2} \\
&= \frac{-2i}{\sqrt{\pi}}t^{-\frac{\alpha}{2}-1}\mathrm{sign}(t)\cos(\alpha\pi/4)\Gamma(1+\alpha/2)
\end{aligned} \tag{282}$$

Inserting this in the second saddle point equation, we obtain

$$\begin{aligned}
&\left[-i\frac{1}{2\pi b}(\frac{(-2i)}{\sqrt{\pi}}t^{\alpha/2-1}\cos(-\alpha\pi/4)\Gamma(1-\alpha/2)\right]^{(}q-1)\mathrm{sign}(t) \\
= \quad & -ib\frac{-2i}{\sqrt{\pi}}t^{-\frac{\alpha}{2}-1}\mathrm{sign}(t)\cos(\alpha\pi/4)\Gamma(1+\alpha/2)
\end{aligned} \tag{283}$$

This gives that $\alpha = 2(1-2\Delta)$ so that

$$G(t) = b\mathrm{sign}(t)/|t|^{2\Delta}. \tag{284}$$

.

The coefficient b is also determined by this equation.

15.7 Diffeomorphism Invariance

The mean field equations are inhomogenous equations, and will have a unique solution subject to boundary conditions. We see immediately that if $G(t,t')$ is a solution than also $G(t+a,t'+a)$ is a solution. So for translational invariant boundary conditions, we have that the solution only depends on the difference of its arguments, ile $G(t,t') = G(t-t')$. This also implies

$$G(t) = \langle\chi(t)\chi(0)\rangle = \langle\chi(0)\chi(-t)\rangle = -\langle\chi(-t)\chi(0)\rangle = -G(-t). \tag{285}$$

That is the reason we choose the Ansatz $G(\tau) \sim \text{sign}(\tau)$.

The saddle point equations have a much larger invariance – they are invariant under diffeomorphism $\tau \to f(\tau)$. We can easily check that if $G(\tau, \tau')$ is a solution than also

$$(f'(\tau)f'(\tau'))^{\Delta} G(f(\tau), f(\tau')). \tag{286}$$

is also a solution. $\Sigma(\tau, \tau')$ is determined by the mean field equation while

$$\begin{aligned} \Sigma \cdot G &= \int G(f(\tau), f(s))\Sigma(f(s), f(\tau''))(f'(\tau))^{\Delta}(f'(\tau''))^{3\Delta} df(s) \\ &= -(f'(\tau))^{\Delta}(f'(\tau''))^{3\Delta}\delta(f(\tau) - f(\tau'')) \\ &= -(f'(\tau))^{\Delta}(f'(\tau''))^{3\Delta}\frac{1}{|f'(\tau)|}\delta(\tau - \tau'') \\ &= -\delta(\tau - \tau''). \end{aligned} \tag{287}$$

We assumed that $f(\tau)$ is an increasing function.

This invariance can be used to construct a solution at finit temperature. A transformation that maps $[0, \beta]$ to the real axis is given by

$$f(\tau) = \tan\frac{\tau\pi}{\beta}. \tag{288}$$

Then

$$f'(\tau) = \frac{\pi}{\beta}\cos^{-2}\frac{\tau\pi}{\beta} \tag{289}$$

So

$$\begin{aligned} G(\tau) &= b\text{sign}(\tau)(\frac{\beta}{\pi})^{2\Delta}\cos^{-2\Delta}\frac{\tau\pi}{\beta}\frac{1}{\tan^{2\Delta}(\tau(\frac{\beta}{\pi})} \\ &= b(\frac{\beta}{\pi})^{\Delta}\frac{1}{\sin^{2\Delta}(\frac{\pi\tau}{\beta})}). \end{aligned} \tag{290}$$

Note that $f'(0) = \pi/\beta$.

Not all diffeomorphism do given a new solution,

$$(f'(\tau)f'(0))^{\Delta}\frac{1}{(f(\tau) - f(0))^{2\Delta}} = \frac{1}{\tau^{2\Delta}} \tag{291}$$

Therefore,

$$\frac{f'(\tau)f'(0)}{(f(\tau) - f(0))^2} = \frac{1}{\tau^2} \tag{292}$$

or

$$\frac{d}{d\tau}\frac{1}{f(\tau) - f(0)}f'(0) = \frac{d}{d\tau}\frac{1}{\tau}. \tag{293}$$

This gives

$$\frac{f'(0)}{f(\tau) - f(0)} = c + \frac{1}{\tau}, \tag{294}$$

which is solved by

$$f(\tau) = f(0) + \frac{c}{f'(0)} + \frac{1}{f'(0)\tau}. \tag{295}$$

Since a translation $\tau \to \tau + a$ also gives a new solution, the full invariance group is given by

$$f(\tau) = f(0) + \frac{c}{f'(0)} + \frac{1}{f'(0)(\tau + a))}. \tag{296}$$

This is Sl(2, R). So the Goldstone manifold is given by Diff(1,R)/Sl(2,R).

The time derivative breaks this invariance and we are now going to look for an effective action with the same pattern of symmetry breaking. We require that the effective partition function has the same transformation properties as the exact partition function.

To get an better idea about the soft mode action, we rewrite the noninvariant part of the action as

$$S = -\frac{1}{2}\int d\tau d\tau' \log\det(1 - \delta(\tau - \tau')\partial_\tau \Sigma^{-1}) \tag{297}$$

and keep only the lowest power in the derivative. We also use that to $G = \Sigma^{-1}$ to this order in the derivatives. We now calculate the effect of the reparameterizations on the actio

$$\begin{aligned} \delta S \qquad &= \qquad \frac{1}{2}\int d\tau d\tau' \delta(\tau - \tau')\partial_\tau (G^f - G) \\ \& = \quad &-\tfrac{1}{2}\textstyle\int d\tau d\tau' \partial\tau \delta(\tau - \tau')(G^f - G) \end{aligned} \tag{298}$$

Because of the δ-function, the integral only picks up contributions for $\tau \approx \tau'$. Near this point the solution of the equations of motion is given by

$$\frac{b}{|\tau - \tau'|^{2\Delta}}. \tag{299}$$

Then

$$G^f - G = b\left[\frac{f'(\tau)f'(\tau')}{(f(\tau) - f(\tau'))^2}\right]^\Delta - \frac{b}{|\tau - \tau'|^{2\Delta}}. \tag{300}$$

We now Taylor expand $f'(\tau')$ in the numerator and $(f(\tau)$ in the denominator.

$$\begin{aligned} f(\tau') &= f(\tau) + (\tau' - \tau)f'(\tau) + \frac{1}{2}(\tau' - \tau)^2 f''(\tau) + \frac{1}{6}(\tau' - \tau)^3 f'''(\tau) \\ f'(\tau') &= f'(\tau) + (\tau' - \tau)f''(\tau) + \frac{1}{2}(\tau' - \tau)^2 f'''(\tau) - \frac{b}{|\tau - \tau'|^{2\Delta}}. \end{aligned} \tag{301}$$

This results in

$$\begin{aligned} G^f - G &= b\left[\frac{1 + (\tau' - \tau)f''(\tau)/f'(\tau) + \frac{1}{2}(\tau' - \tau)^2 f'''(\tau)/f'(\tau)}{((\tau - \tau')^2(1 + \frac{1}{2}(\tau' - \tau)f''(\tau) + \frac{1}{6}(\tau' - \tau)^2 f'''(\tau))^2}\right]^\Delta - \frac{b}{|\tau - \tau'|^{2\Delta}} \\ &= b\left[\frac{1}{(\tau - \tau')^2}(1 + ((\tau - \tau')^2\frac{1}{6}f'''(\tau)/f'(\tau) - (f''(\tau)/f'(\tau))^2)\right]^\Delta - \frac{b}{|\tau - \tau'|^{2\Delta}} \\ &= \frac{b}{6}|\tau - \tau'|^{2-2\Delta}\Delta\left(\frac{f'''(\tau)}{f}'(\tau) - \frac{3}{2}\left[\frac{f'''(\tau)}{f}'(\tau)\right]^2\right) \end{aligned}$$

$$\frac{b}{6}|\tau - \tau'|^{2-2\Delta}\Delta\{f, \tau\}, \tag{302}$$

where the expression between the curly brackets is the Schwartzian action. We thus find the action

$$\delta S = -\frac{b}{12}\int d\tau d\tau' \partial\tau \delta(\tau-\tau')|\tau-\tau'|^{2-2\Delta}\Delta\{f,\tau\}. \tag{303}$$

Using $s=\tau'-\tau$ and τ as new integration variables we obtain

$$\begin{aligned}\delta S &= \frac{b\Delta}{12}\int d\tau ds \partial_s\delta(s)|s|^{2-2\Delta}\{f,\tau\} \\ &= \frac{b\Delta}{12}\int d\tau\{f,\tau\}\int ds\partial_s\delta(s)|s|^{2-2\Delta}.\end{aligned} \tag{304}$$

To evaluate the constant, the delta function has to be regularized [**?**].

15.8 Large q expansion

In this section we discuss the large q expansion of the SYK model. In the action, q occurs as

$$\frac{1}{q}G^q\Sigma. \tag{305}$$

For $q\to\infty$ this interaction term is suppressed and we can expand G about the free propagator When we write

$$G=\frac{1}{2}\text{sign}(\tau)(1+\frac{g(\tau)}{q}). \tag{306}$$

Then

$$\begin{aligned}\Sigma &= J^2G^{q-1} = J^2 2^{1-q}\text{sign}(\tau)(1+\frac{g(\tau)}{q})^{q-1} \\ &= J^2 2^{-q}\text{sign}(\tau)e^{g(\tau)}\end{aligned} \tag{307}$$

The constant b is given by

$$J^2 b^q\pi = (\frac{1}{2}-\frac{1}{q})\tan\frac{\pi}{q}\approx\frac{\pi}{2q}. \tag{308}$$

We have that

$$G(\omega)=\frac{1}{-i\omega}+\frac{1}{2q}[\text{sign}g](\omega). \tag{309}$$

so

$$\frac{1}{G(\omega)} = -i\omega+\frac{\omega^2}{q}[\text{sign}g]\omega = -i\omega-\Sigma(\omega). \tag{310}$$

Fourier transforming back we obtain

$$-\partial_\tau^2[sign(\tau)g(\tau)] = -2qJ^2 2^{1-q}\text{sign}\tau e^{g(\tau)}. \tag{311}$$

This differential equation can be solved

$$e^{g(\tau)} = \frac{c^2}{\mathcal{J}^2}\frac{1}{\sin^2(c(|\tau|+\tau_0)}} \tag{312}$$

At very short times we should recover the free fermion result, so

$$g(0) = g(\beta) = 0. \tag{313}$$

The solution that satisfies these boundary conditions is given by

$$e^{g(\tau)} = \frac{\cos^2 \pi v}{\cos^2(\pi v/2 - |\tau|/\beta)}, \qquad \cos \pi v/2 = c^2/\mathcal{J}^2 \tag{314}$$

The second equation comes from noticing that $c = \pi v/\beta$ and $c^2/\mathcal{J}^2 = \cos^2 \pi v/2$.

Next we calculate the action of the large-q solution. Because of the deivative we calculate $\mathcal{J}\partial_{\mathcal{J}} \log Z$ and notice that because the partition function only depends on the combination $\beta\mathcal{J}$ we have that

$$\mathcal{J}\partial_{\mathcal{J}} \log Z = \beta \partial_\beta \log Z. \tag{315}$$

To obtain the free energy, we have to integrate this equation.

$$\begin{aligned} \mathcal{J}\partial_{\mathcal{J}} \log Z &= \mathcal{J}\partial_{\mathcal{J}}(-\beta F) = \frac{\beta \mathcal{J}^2}{2q^2} \int_0^\beta d\tau e^{g(\tau)} \\ &= \frac{\beta \mathcal{J}^2}{2q^2} \frac{\beta}{\pi v} \cos^2 \frac{\pi v}{2} \int_0^\beta d\tau \frac{d}{d\tau} \tan(\pi v(1/2 - |\tau|/\beta)) \\ &= \frac{\beta \mathcal{J}^2}{q^2} \frac{\beta}{\pi v} \cos^2 \frac{\pi v}{2} \tan(\pi v/2) \end{aligned} \tag{316}$$

We now use the relation between v and $\beta\mathcal{J}$ to obtain

$$\frac{d\beta\mathcal{J}}{\beta\mathcal{J}} = \frac{dv}{v} + dv \frac{\pi}{2} \tan \frac{\pi v}{2}. \tag{317}$$

This results in

$$\frac{v}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}} \partial v(-\beta F) = N \frac{\pi v}{q^2} \tan \frac{\pi v}{2}. \tag{318}$$

The boundary condition of this differential equation is given by the condition that for $v \to 0$ the zero coupling limit should be recovered. Then

$$-\beta F|_{J=0} = N \frac{1}{2} \log 2. \tag{319}$$

This results in the solution

$$-\frac{\beta F}{N} = \frac{1}{2} \log 2 + \frac{\pi v}{q^2}(tan \frac{\pi v}{2} - \frac{\pi v}{4}). \tag{320}$$

We can calculate the entropy in the low temperature expansion. For $\beta \to \infty$ we have that $v \to 1$. If we expand $v = 1 + \delta v$ then

$$-\pi \delta v 2 = \frac{\pi}{\beta \mathcal{J}}. \tag{321}$$

and

$$-\frac{\beta F}{N} = \frac{1}{2} \log 2 - \frac{\pi^2}{4q^2} + \frac{\beta \mathcal{J}}{q^2}. \tag{322}$$

We find the remarkable result that the zero temperature entropy is extensive.

16 Spectral Density of the SYK Model

16.1 Moments of the Hamiltonian

The moments of the Hamiltonian are given by

$$\langle M_{2p} = \frac{1}{L}\mathrm{Tr}H^{2p}\rangle, \tag{323}$$

where the average is over the Gaussian distributed couplings. A Gaussian integral is given by the sum over all Wick contractions. If all Wick contraction are independent and the terms in the Hamiltonian commute, and using the notation

$$H = \sum J_\alpha \Gamma_\alpha. \tag{324}$$

we obtain

$$\frac{1}{L}\mathrm{Tr}H^{2p} = (2p-1)!![\sum_\alpha \langle J_\alpha J_\alpha\rangle]^p, \tag{325}$$

where we have used that $\Gamma_\alpha^2 = 1$. We thus find

$$M_{2p} = (2p-1)!!M_2^2 \tag{326}$$

with

$$M_2 = \binom{N}{q}\sigma^2. \tag{327}$$

These are the moments of a Gaussian distribution. This is a good approximation for $N \gg q$ when most of the terms in the Hamiltonian commute.

Next we use that a crossing a two contraction lines results in the factor

$$\eta = 2^{-N/2}\binom{N}{q}^{-1}\sum_\beta \mathrm{Tr}[\Gamma_\alpha, \Gamma_\beta] = \binom{N}{q}^{-1}\sum_{p=0}^{q}(-1)^p\binom{q}{p}\binom{N-q}{q-p}. \tag{328}$$

When we treat all contractions as independent, the moments are thus given by

$$M_{2p} = M_2^p \sum_k C_k \eta^k, \tag{329}$$

where C_k is the number of diagrams with k crossings. This sum is known as the Riordan-Touchard formula

$$\sum_k C_k\eta^k = \frac{1}{(1-\eta)^p}\sum_{k=-p}^{p}(-1)^k\eta^{k(k-1)/2}\binom{2p}{p+k}. \tag{330}$$

These are the moments of the weight function of the Q-Hermite polynomials. The spectral density is thus given by this weight function

$$\rho^{QH}(E) = \sqrt{1-(E/E_0)^2}\prod k = 1^\infty\left[1 - 4\frac{E^2}{E_0^2}\frac{1}{2+\eta^k+\eta^{-k}}\right] \tag{331}$$

with

$$E_0^2 = \frac{4\sigma^2}{1-\eta}. \tag{332}$$

Since the Q-Hermite approximation work so well, we can expand the spectral density as

$$\rho(E) = \rho^{QH}(E)(1 + \sum_k a_k H_k^\eta(E/E_0)) \tag{333}$$

The Q-Hermite polynomials satisfy the recursion relation [**?**]

$$H_{n+1}^\eta(x) = xH_n^\eta(x) - \sum_{k=0}^{n-1} \eta^k H_{n-1}^\eta(x) \tag{334}$$

with

$$H_0^\eta(x) = 1 \qquad \text{and} \qquad H_1^\eta(x) = x. \tag{335}$$

The orthogonality relations are given by

$$\int_{-E_0}^{E_0} dx \rho_{QH}(x) H_n^\eta(x) H_m^\eta(x) = \delta_{nm} n_\eta!, \tag{336}$$

where $p_\eta!$ is the Q-factorial defined as

$$n_\eta! = \prod_{k=1}^{n-1} (\sum_{s=0}^{k} (1+\eta^s). \tag{337}$$

This expansion is a generalization of the Gram-Charlier expansion – for $\eta = 1$ it becomes the Gram-Charlier expansion.

17 The Maldacena-Qi Model

The Maldacena-Qi model is a model for an eternal wormhole. It is defined by two SYK models coupled by an interaction. The fermions in each SYK model anti-commute. Specifially, we have

$$H = H^L + H^R + kS \tag{338}$$

with

$$H^L = \sum_{abcd} J_{abcd} \chi_a^L \chi_b^L \chi_c^L \chi_d^L, \qquad H^R = \sum_{abcd} J_{abcd} \chi_a^R \chi_b^R \chi_c^R \chi_d^R, \qquad . \tag{339}$$

and the interacting term which we will refer to as the spin operator, is given by

$$S = ik \sum_n \chi_n^L \chi_n^R \tag{340}$$

The coupling term J_{abcd} of the left and right Hamiltonian are the same.

For $k = 0$ the eigenstates of the model are given by the tensor product of the eigenstates of H^L and H^R,

$$H^L|m\rangle = E_m|m\rangle, \qquad H^R|m\rangle = E_m|m\rangle. \tag{341}$$

Then the eigenvalues of the unperturbed Hamiltonian are given by

$$(H^L + H_R)|m\rangle \otimes |n\rangle = (E_m + E_n)|m\rangle \otimes |n\rangle \tag{342}$$

We can also easily find the eigenstates of the spin operator by noting that each of the terms, the product of two gamma matrices, commute. Therefore, they can be diagonalized simultaneously. Since each term squares to $1/4$, the eigenvaluse of each term are are ± 12. The energy of the ground state is thus $-N/4$, and the energy increases in steps of 1 until $N/4$ for the excited states. The degeneracy of the eigenvalue $-N/4 + p$ is given by

We can also prove this formally by introducing the raising and lowering operators

$$\Gamma_k^{\pm} = \Gamma_k^L \pm i\Gamma_k^R. \tag{343}$$

Then

$$[S, \Gamma_k^{\pm}] = [i\Gamma_k^R\Gamma_k^L, \Gamma_k^{\pm}] = \Gamma_k^{\pm}. \tag{344}$$

This implies that if

$$S|\phi\rangle = s|\phi\rangle \tag{345}$$

then

$$S\Gamma^+|\phi\rangle = (\Gamma^+ S + \Gamma^+)|\phi\rangle = (s+1)\Gamma^+|\phi\rangle. \tag{346}$$

We can also write the unperturbed Hamiltonian in terms of $\Gamma^{\pm}$

$$\begin{aligned}\Gamma_a^L\Gamma_b^L\Gamma_c^L\Gamma_d^L + \Gamma_a^R\Gamma_b^R\Gamma_c^R\Gamma_d^R &= \frac{1}{16}\left[(\Gamma_a^+ + \Gamma_a^-)(\Gamma_b^+ + \Gamma_b^-)(\Gamma_b^+ + \Gamma_b^-)(\Gamma_b^+ + \Gamma_b^-)\right. \\ &\left. +(\Gamma_a^+ - \Gamma_a^-)(\Gamma_b^+\Gamma_b^-)(\Gamma_b^+ - \Gamma_b^-)(\Gamma_b^+ - \Gamma_b^-)\right].\end{aligned} \tag{347}$$

It is clear that the terms with an odd number of $\Gamma^{\pm}$ matrices vanish. Therefore, $H^R + H^L$ raises or lowers the spin by four units or does not change the spin. This implies that the Hamiltonian has an S mod 4 symmetry

$$[e^{i\frac{\pi}{2}S}, H] = 0. \tag{348}$$

17.1 The Thermo Field Double State

This state is referred to as the TFD. Let us first look at the ground state of the spin operator. We already know that the eigenvalue is equal to $-N/4$. Let us now choose the following gamma matrices

$$\begin{aligned}\Gamma_k^L &= \Gamma_k \otimes 1, \\ \Gamma_k^R &= \Gamma_5 \otimes \Gamma_k,\end{aligned} \tag{349}$$

and the Γ_k are random matrices in $N/2$ dimensions. Let us now consider the expectation value of S in the state

$$|gs\rangle = c\sum |m\rangle \otimes AK|m\rangle \tag{350}$$

with $|m\rangle$ a complete set of states in the space of $N/2$ Majoranas which are also eigenstates of Γ_5,

$$\Gamma_5|m\rangle = \chi_m|m\rangle, \tag{351}$$

and A is an operator that is yet to be determined. The constant c is a normalization constant, $c = 2^{-N/8}$, which normalizes the state to 1. Then

$$\begin{aligned}
\langle gs|S|gs\rangle &= ic^2\sum_k\sum_{mn}\langle m|\Gamma_k\Gamma_5|n\rangle\langle Km|A^\dagger\Gamma_k A|Kn\langle \\
&= ic^2\sum_k\sum_{mn}\chi_n\langle m|\Gamma_k|n\rangle\langle n|(A^\dagger\Gamma_k A)^T|n\langle \\
&= ic^2\sum_k\sum_{mn}\chi_n\langle m|\Gamma_k|n\rangle\langle n|A^T\Gamma_k^* A^*|n\langle \\
&= ic^2\sum_k\sum_{mn}\chi_n\langle m|\Gamma_k|n\rangle\langle n|A^T C\Gamma_k C^{-1}A^*|n\langle .
\end{aligned}$$

We have to produce a χ_n and eliminate C so that we can use completeness and $\Gamma_k^2 = 1/2$. Let us try

$$A = e^{i\frac{\pi}{4}\Gamma_5}C. \tag{352}$$

Then,

$$\begin{aligned}
\langle gs|S|gs\rangle &= ic^2\sum_k\sum_{mn}\chi_n\langle m|\Gamma_k|n\rangle\langle n|(-C)e^{i\frac{\pi}{4}\Gamma_5}C\Gamma_k C^{-1}C^* e^{-i\frac{\pi}{4}\Gamma_5}C^*|n\langle \\
&= ic^2\sum_k\sum_{mn}\chi_n i\chi_n\langle m|\Gamma_k|n\rangle\langle n|(-C)C\Gamma_k C^{-1}C^*|n\langle .
\end{aligned}$$

We now can use completeness and the fact that $\Gamma_k^2 = 1/2$ which shows that the expectation value is $-N/4$. Since the ground state is nondegenarate, we have that it is given by

$$|gs\rangle = 2^{-N/4}\sum |m\rangle \otimes e^{i\frac{\pi}{4}\Gamma_5}CK|m\rangle \tag{353}$$

We are free to choose the states $|m\rangle$ other than they are eigenstates of Γ_5. We now choose $|m\rangle$ the eigenstates of H^R. Since

$$[e^{\frac{\pi i}{4}\Gamma_5}CK, H^R] = 0 \tag{354}$$

we also choose the states $|m\rangle$ to be eigenstates of $e^{\frac{\pi i}{4}\Gamma_5}CK$.

The TFD is now defined as

$$\begin{aligned}
|TFD\rangle &= e^{-\beta(H^R+H^L)}|gs\rangle \\
&= 2^{-N/4}\sum e^{-\beta 2E_m}|m\rangle \otimes e^{i\frac{\pi}{4}\Gamma_5}CK|m\rangle
\end{aligned} \tag{355}$$

The claim is that this state is a good approximation to the ground state of the coupled SYK model at $k \neq 0$. The inverse temperature β is a parameter that can be obtained by minimizing the energy of the state.

17.2 Anti-Unitary Symmetry

In a standard representation, half of the γ matrices are real and the other half is purely imaginary. Therefore we can choose the left γ matrices real and the right γ matrices purely imaginary. Then the spin operator is real and H^R and H^L, both the sum of the product of q γ matrices, are real as well. So the Hamiltonian is real, and the levels will be correlated according to the GOE in a given S-mod 4 sector if the system is chaotic.

17.3 S mod 4 Symmetry

Let us do the explicit construction of the γ matrices. In we know the γ matrices in dimension $2(d-1)$, then two γ matrices are new

$$\begin{aligned} \Gamma_2^{2d} &= \sigma_2 \overbrace{\otimes 1 \cdots \otimes 1}^{d-1} \\ \Gamma^{2d} d_1 &= \sigma_1 \otimes \sigma_3 \overbrace{\otimes 1 \cdots \otimes 1}^{d-2} \end{aligned} \tag{356}$$

The other γ matrices are contructed recursively

$$\Gamma_{k+2}^{2d} = \sigma_1 \otimes \Gamma_k^{2d-2} \tag{357}$$

So the odd Γ matrices are real while the even Γ matrices are purely imaginary. Clearly

$$\Gamma_1^{2d} \Gamma_2^{2d} = i\sigma_3 \otimes \sigma_3 \overbrace{\otimes 1 \cdots \otimes 1}^{d-2} \tag{358}$$

is diagonal with eigenvalues $\pm i$. The other products of even and odd Γ matrices are diagonal recursively. Let us look at $2d = 4$. Then

$$\begin{aligned} \Gamma_1 &= \sigma_1 \otimes \sigma_3 \\ \Gamma_2 &= \sigma_2 \otimes 1 \\ \Gamma_3 &= \sigma_1 \otimes \sigma_1 \\ \Gamma_4 &= \sigma_1 \otimes \sigma_2. \end{aligned} \tag{359}$$

We have thus shown that the spin operator is diagonal with eigenvalues given by

$$-\frac{N}{4} + p, \quad p = 0, \cdots \frac{N}{2}. \tag{360}$$

The $Smod4$ symmetry operator is thus given by

$$e^{\frac{\pi i}{2}(p - \frac{N}{4})}. \tag{361}$$

Ignoring the overall factor $\exp(\pi i N/8)$, the eigenvalues are given $\pm i$ and ± 1. Since this operator commutes with the Hamiltonian, the Hamiltonian splits into four blocks.

18 Hawking-Page Phase Transition

The gound state of the system is interpreted as an eternal wormhole with thermal excitations at low temperatures. At a critical temperature the system turns into a two black hole phase. The first order phase transition between these thwo phases is known as the Hawking-Page transition.

We can calculate the partition function of the spin operator,

$$Z = \sum_p \binom{N/2}{p} e^{-\beta k(-\frac{N}{4}+p)} = e^{\beta k \frac{N}{4}}(1+e^{-\beta k})^{N/2}. \tag{362}$$

This results in the free energy

$$-\frac{\beta F}{N} = -\frac{\beta k}{4} + \frac{1}{2}\log(1+e^{-\beta k}) \tag{363}$$

This is an integrable model and obviously, it cannot have a phase transition. However there is a crossover at $\beta = 1/k$, which is the temperature difference between the ground state and the first excited state.

This phase transition can also be studied in the ΣG formulation of the SYK model. The action is given by [**?**]

$$\begin{aligned} -\frac{S}{N} &= \log Pf(\partial_\tau \delta_{ab} - \Sigma_{ab}) - \frac{1}{2}\int d\tau_1 \tau_2 \sum_{ab}\left(\Sigma_{ab}(\tau_1,\tau_2)G_{ab}(\tau_1-\tau_2) - s_{ab}\frac{\mathcal{J}^2}{2q^2}(2G_{ab}(\tau_1,\tau_2))^q\right) \\ &\quad + \frac{ik}{2}\int d\tau[-G_{LR}(\tau) + G_{RL}(\tau] \end{aligned} \tag{364}$$

The sum is over L and R with $s_{LL} = s_{RR} = 1$ and $s_{LR} = s_{RL} = (-1)^{q/2}$. We can again derive the Schwinger-Dyson equations which can be solved numerically.

19 Special Topics

- Calculation of OTOC in SYK Model, see Maldacena and Stanford
- Nonlinear σ-model for SYK, Altland and Bagrets
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- Random Matrix Theory for QCD at nonzero chemical potential
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